

Amplification of Slow Magnetosonic Waves by Shear Flow: Heating and Friction Mechanisms of Accretion Disks *

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Propagation of three dimensional magnetosonic waves is considered for a homogeneous shear flow of an incompressible fluid. The analytical solutions for all magnetohydrodynamic variables are presented by confluent Heun functions. The problem is reduced to finding a solution of an effective Schrödinger equation. The amplification of slow magnetosonic waves is analyzed in great details. A simple formula for the amplification coefficient is derived. The velocity shear primarily affects the incompressible limit of slow magnetosonic waves. The amplification is very strong for slow magnetosonic waves in the long-wavelength limit. It is demonstrated that the amplification of those waves leads to amplification of turbulence. The phenomenology of Shakura–Sunyaev for the friction in accretion disks is derived in the framework of the Kolmogorov turbulence. The presented findings may be the key to explaining the anomalous plasma heating responsible for the luminosity of quasars. It is suggested that wave amplification is the keystone of the self-sustained turbulence in accretion disks.

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I. INTRODUCTION

The occurrence of intense dissipation in accretion flows is among the long-standing unsolved problems in astrophysics.^{1,2} Revealing how the magnetized turbulence creates shear stress tensor is of primary importance to understand the heating mechanism and the transport of angular momentum in accretion disks. The transport of angular momentum at greatly enhanced rates is important for the main problem of cosmogony, that is understanding the dynamics of creation of compact astrophysical objects.^{3,4} Without a theory explaining the enhanced energy dissipation in accretion flows of turbulent magnetized plasma we would have no clear picture of how our solar system has been created, why the angular momentum of the Sun is only 2% of the angular momentum of solar system, while carrying 99% of the solar system’s mass, why quasars are the most luminous sources in the universe. The importance of friction forces and convection as well the problem of angular momentum redistribution for the first time was emphasized by von Weizsäcker;⁵ a very detailed bibliography on the physics of disks is provided in the monograph by Morozov and Khoperskov.⁶ Gravitational forces, angular momentum conservation, and dissipation processes become the main ingredients of the standard model of accretion disks by Shakura and Sunyaev⁷ and Lynden-Bell and Pringle.⁸ Lynden-Bell⁹ suggested that quasars are accretion disks and Shakura and Sunyaev introduced alpha phenomenology for the stress tensor

$$\sigma_{R\varphi} = \alpha p, \quad p \sim \rho c_s^2, \quad (1)$$

where α is a dimensionless parameter, p is the pressure ρ is the mass density, c_s is the sound speed, and the indices of the tensor come from the cylindrical coordinate system (R, φ, z) related to the disk rotating around the z -axis. As accretion disks can have completely different scales for protostellar disks, mass transfer disks, and disks in active galactic nuclei (AGN) it is unlikely that Coriolis force is the main cause for dissipation, while the bending of the trajectories is a critical ingredient. We suppose that the shear dissipation in magnetized turbulent plasma is a robust and very general phenomenon which can be analyzed as a local heating for approximately homogeneous magnetic field and gradient of the velocity. For these reasons in the next section we will use local Cartesian coordinates choosing the x -axis in radial direction $\mathbf{e}_x \equiv \hat{R}$, and the y -axis along the circulation of the almost Keplerian motion of accreting plasma, $\mathbf{e}_y \equiv \hat{\varphi}$. The axis of the disk is along $\mathbf{e}_z \equiv \hat{z}$.

There is almost a consensus that the magnetic field is essential and should be introduced from the very beginning in the magnetohydrodynamic (MHD) analysis. The likely importance of MHD waves on the cosmogony of solar system

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has been pointed out by Alfvén: “At last some remarks are made about the transfer of momentum from the Sun to the planets, which is fundamental to the theory. The importance of the magnetohydrodynamic waves in this respect is pointed out.¹⁰” As the magnetic field lines are frozen in the highly conducting hot plasma, we suppose that they lie in the plane of the disk $\mathbf{B} \approx -B\mathbf{e}_\varphi$. In our local analysis we can suppose that mass density is also constant $\rho \approx \text{const}$. Moreover, we assume that the magnetic field is sufficiently small and the Alfvén speed is much smaller than the sound speed, $V_A \ll c_s$. As usual, the magnetic pressure is defined as

$$\frac{1}{2}\rho V_A^2 = p_B \equiv \frac{B^2}{2\mu_0}. \quad (2)$$

Under these conditions, we can consider the plasma as an incompressible fluid. In other words we assume a plasma for which the magnetic pressure p_B is much smaller than the gaseous one p

$$\beta \equiv \frac{p}{p_B} \gg 1. \quad (3)$$

According to the theory of turbulent dynamo the density of magnetic energy p_B is comparable with the density of kinetic energy of the turbulent motion at a basic scale which is smaller than the pressure, i.e., we suppose that the turbulence creating the magnetic field is subsonic.

Our goal is to demonstrate that small magnetic fields can catalyze a disk to ignite as a star and to help the kinetic energy of the shear flow to be converted into heat in spite of the very low molecular plasma viscosity η .

In a weak magnetic field the incompressible transverse MHD waves, the slow magnetosonic waves (SMWs) and Alfvén waves (AWs), are qualitatively new features of the weakly magnetized plasma. Under these conditions every mechanism of giant wave amplification inevitably acts as a dissipation mechanism for heat production which creates stress tensor and a transport of angular momentum in accretion disks.

The present research is triggered by the amplification of SMWs observed by numerical analysis of two-dimensional (2D) MHD waves.¹¹ Even these first investigations demonstrated that shear flow leads to exchange of energy and wave amplification, mutual transformation between different wave modes, gave the perspectives to explain self-heating and other processes now known as *nonmodal*. As we have standing wave amplification often is used the notion *overreflection*; a running wave with k_y is amplified together with the reflected wave with wave-vector $-k_y$. These conclusions was confirmed extended and popularized by the numerical investigations.¹² However without a realistic analytical three-dimensional solution the investigations of shear flows is in state of infancy. The purpose of our work is to analytically solve the simplest case of three-dimensional (3D) waves in homogeneous shear flow and magnetic field, to analytically calculate the phase averaged amplification coefficient \mathcal{A} of the waves and to incorporate this waves’ amplification into some standard model for turbulence. Or, in short, our final aim is to use an exact solution for linearized waves in order to build up a coherent scenario for accretion disk theory and luminosity of quasars. Before continuing we wish to refer to the concluding remarks from the review by Balbus and Hawley:³ *Again and again we are ignorant. The good news is that, for first time, it appears that we know in which directions we should be looking to begin to find answers to questions like these.*

II. MHD MODEL. ANALYTICAL SOLUTION TO LINEARIZED WAVE EQUATIONS

Our model problem is to investigate MHD waves in a homogeneous shear flow and a homogeneous magnetic field

$$\mathbf{V}_{\text{shear}} = Ax\mathbf{e}_y, \quad \mathbf{B}_0 = -B_0\mathbf{e}_y, \quad \mathbf{e}_i = \partial_i\mathbf{r}, \quad \mathbf{e}_B \equiv \mathbf{B}_0/B_0 = -\mathbf{e}_y, \quad (4)$$

$$= \nabla \times \mathbf{V}_{\text{shear}} = A\mathbf{e}_z, \quad i = 1, 2, 3, \quad \mathbf{e}_\Omega = \Omega/\Omega = \mathbf{e}_z \quad (5)$$

in an ideal and incompressible plasma. For the linearized MHD equations we suppose a time dependent wave-vector

$$\mathbf{k}(t) = -Atk_y\mathbf{e}_x + k_y\mathbf{e}_y + k_z\mathbf{e}_z, \quad \mathbf{k} \cdot \mathbf{r} = (y - Atx)k_y + k_z z \quad (6)$$

in the plane waves solution.¹⁴ For the velocity \mathbf{V} and the magnetic field \mathbf{B} we consider small enough time-dependent amplitudes \mathbf{v} and \mathbf{b}

$$\begin{aligned} \mathbf{V} &= \mathbf{V}_{\text{shear}} + \mathbf{V}_{\text{wave}}, & \mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_{\text{wave}}, \\ \mathbf{V}_{\text{wave}} &= -iV_A\mathbf{v}(t)e^{i\mathbf{k}\cdot\mathbf{r}}, & \mathbf{B}_{\text{wave}} &= B_0\mathbf{b}(t)e^{i\mathbf{k}\cdot\mathbf{r}}. \end{aligned} \quad (7)$$

The magnetic field \mathbf{B} and velocity \mathbf{V} are, of course, real vector fields as functions of t and the position vector $\mathbf{r} = (x, y, z)$, but it is mathematically convenient to use complex variables and calculate the real parts of the derived solutions.

Let us summarize the supposed approximations: incompressible $\nabla \cdot \mathbf{V} = 0$, and inviscid $\eta = 0$, fluid with constant density $\rho = \text{const}$, and negligible Ohmic resistivity $\varrho = 0$. For a sufficiently slow nonrelativistic motion the MHD equations take the form¹⁷

$$\rho D_t \mathbf{V} = -\nabla p - \mathbf{B} \times (\nabla \times \mathbf{B})/\mu_0, \quad D_t \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{V}, \quad (8)$$

where $D_t \equiv \partial_t + \mathbf{V} \cdot \nabla$ is the substantial Lagrange derivative; $\nabla \cdot \mathbf{B} = 0$.

Let us introduce the characteristic length-scale Λ and velocity V_A of the problem

$$\Lambda \equiv \frac{V_A}{A} \ll R, \quad \mathbf{V}_A \equiv \frac{\mathbf{B}_0}{\sqrt{\mu_0 \rho}}, \quad V_A \equiv |\mathbf{V}_A| \ll c_s, \quad (9)$$

dimensionless wave-vector $\mathbf{K} \equiv \Lambda \mathbf{k}$, and dimensionless time

$$\tau \equiv At = -\frac{K_x(t)}{K_y} = -\frac{k_x(t)}{k_y}. \quad (10)$$

The initial time is chosen so that $k_x(t=0) = 0$. We suppose that the characteristic length Λ is much smaller than the disk thickness d_{disk} .

As the shear parameter describes the radial gradient of the orbiting velocity

$$A \equiv \frac{\partial V_\varphi}{\partial R} = -\frac{1}{2}\omega_{\text{Kepler}} \quad (11)$$

at the corresponding radius R , it is of order of the angular frequency ω_{Kepler} of the Keplerian motion. We suppose also that for weak magnetic fields the Alfvén speed is much smaller than the velocity of Keplerian orbiting, $V_A \ll V_{\text{Kepler}} \equiv R\omega_{\text{Kepler}}$. In this case the wavelength of the slow magnetosonic waves is much smaller than the radius of the orbit $R\sqrt{k_\varphi^2 + k_z^2} \gg 1$ and the disk thickness $|k_z|d_{\text{disk}} \gg 1$ and we approximately consider the shear flow as homogeneous. In other words, the goal of the present research is to investigate the statistical properties of a shear flow determined by short wavelength waves inaccessible by direct astronomical observations and usual computer simulations of accretion disk magnetohydrodynamics.

The Coriolis force has a negligible influence on the propagation of magnetohydrodynamic waves and their amplification analyzed in the present work for the case of dominating magnetic field. Consequently, in our further consideration we will neglect that force. In such a way we have to investigate the influence of a homogeneous shear flow on small amplitude MHD waves in an incompressible ideal fluid. The homogeneous magnetic field we suppose to be parallel to the shear flow.

With so introduced variables the substitution of wave components from Eq. (7) in MHD equations Eq. (8) after some algebra yields¹¹

$$d_\tau v_x = 2\frac{K_y K_x}{K^2} v_x - K_y b_x, \quad d_\tau b_x = K_y v_x, \quad (12)$$

$$d_\tau v_z = 2\frac{K_y K_z}{K^2} v_x - K_y b_z, \quad d_\tau b_z = K_y v_z, \quad (13)$$

$$\mathbf{K} \cdot \mathbf{v} = 0 = \mathbf{K} \cdot \mathbf{b}, \quad \mathbf{K} = -K_y \tau \mathbf{e}_x + K_y \mathbf{e}_y + K_z \mathbf{e}_z. \quad (14)$$

An unabridged derivation of this set of equations is given in Ref. [15]. Later on, the substitution of $d_\tau v_x$ with $d_\tau^2 b_x$ in Eq. (12), and analogous substitution of $d_\tau v_z$ in Eq. (13) leads to

$$d_\xi^2 b_x + \frac{2\xi}{1+\xi^2} d_\xi b_x + Q^2 b_x = 0, \quad (15)$$

$$d_\xi^2 b_z + Q^2 b_z = 2K_z \frac{v_x}{1+\xi^2}, \quad (16)$$

where

$$\xi \equiv \tau \frac{K_y}{Q} = -\frac{K_x}{\sqrt{K_y^2 + K_z^2}}, \quad (17)$$

$$Q \equiv \sqrt{K_y^2 + K_z^2} = \text{const}. \quad (18)$$

For a two-dimensional motion ($K_z = 0$) in the plane of the disk $\xi = \tau$. The negative friction for $\xi < 0$ in the oscillator equation Eq. (15) comes from the time-dependence of the x -component of the wave-vector

$$K_x(\tau) = -K_y\tau = -Q\xi. \quad (19)$$

The other components are constant $K_y = \text{const}$, and $K_z = \text{const}$. In such a way the change of the sign in the effective friction force is a property of the shear flow related to the change of the sign of the corresponding component of the wave-vector.

For the Alfvén wave amplitude b_z in Eq. (16) we have the equation of a harmonic oscillator with an external force. This equation has the Green function

$$(\text{d}_\xi^2 + Q^2) G(\xi, \xi') = \delta(\xi - \xi') = \text{d}_\xi \theta(\xi - \xi'), \quad (20)$$

$$G(\xi, \xi') = \frac{\sin[Q(\xi - \xi')]}{Q} \theta(\xi - \xi'), \quad (21)$$

and the general solution to Eq. (16) reads

$$\begin{aligned} b_z(\xi) &= \frac{2K_z}{Q} \int_{-\infty}^{\xi} \sin[Q(\xi - \xi')] \frac{v_x(\xi')}{1 + \xi'^2} \text{d}\xi' + \chi(\xi), \\ (\text{d}_\xi^2 + Q^2) \chi(\xi) &= 0, \quad \chi(\xi) = \tilde{C}_g \chi_g + \tilde{C}_u \chi_u, \\ \chi_g(\xi) &= \cos(Q\xi), \quad \chi_u(\xi) = \frac{\sin(Q\xi)}{Q}, \end{aligned} \quad (22)$$

where \tilde{C}_g and \tilde{C}_u are the amplitudes the even and the odd eigen χ -oscillations related to the pure Alfvén waves (AWs) for the case of $K_z = 0$. AWs are not influenced by the shear flow. For the solution of the Cauchy problem with fixed $b_z(\xi_0)$ and $\text{d}_\xi b_z(\xi_0) = Qv_z(0)$ we have

$$b_z(\xi) = \frac{2K_z}{Q} \int_{\xi_0}^{\xi} \sin[Q(\xi - \xi')] \frac{v_x(\xi')}{1 + \xi'^2} \text{d}\xi' + b_z(\xi_0) \cos[Q(\xi - \xi_0)] + \text{d}_\xi b_z(\xi_0) \frac{\sin[Q(\xi - \xi_0)]}{Q}. \quad (23)$$

For the special case of $K_z = 0$ we have a complete separation of variables in two independent sets of equations. One of them is for Alfvén waves with oscillating b_z and v_z perpendicular to the wave-vector and magnetic field, and the second set is for the independent slow magnetosonic waves (SMWs) for which the oscillations of the magnetic field and velocity are in the plane of the external magnetic field and the wave-vector. Still we can use the terminology AWs and SMWs for waves having nonzero K_z . In this case the amplitude of the SMW b_x plays the role of external driving force for the amplitude of AW in Eq. (16). We will see later that amplification is related to SMWs, and AWs with b_z components are only influenced by SMWs at nonzero K_z . Due to a resonance this influence, however, could be significant. It is notable that AW amplitude b_z has provides no feedback on the SMW amplitude b_x .

MHD equations give the possibility to express the velocity by the magnetic field. For the velocity from Eq. (12) and Eq. (13) we have

$$v_z(\xi) = \frac{\text{d}_\xi b_z(\xi)}{Q}, \quad v_x(\xi) = \frac{\text{d}_\xi b_x(\xi)}{Q}, \quad \text{d}_\xi \equiv \frac{Q}{K_y} \text{d}_\tau. \quad (24)$$

The incompressibility conditions Eq. (14) yield explicit expressions for the oscillations parallel to the external magnetic field and shear flow

$$b_y = -\frac{K_x b_x + K_z b_z}{K_y} = -\frac{-K_y \tau b_x + K_z b_z}{K_y} = \frac{Q}{K_y} \xi b_x - \frac{K_x}{K_y} b_z, \quad v_y = -\frac{K_x v_x + K_z v_z}{K_y} = \frac{Q}{K_y} \xi v_x - \frac{K_x}{K_y} v_z. \quad (25)$$

In such a way the main detail in solving the set of Eq. (12) and Eq. (13) is to derive an analytic formula for b_x . The first derivative $\text{d}_\xi b_x$ term in Eq. (15) disappears by the substitution

$$b_x(\xi) = \frac{\psi(\xi)}{\sqrt{1 + \xi^2}}, \quad (26)$$

and we arrive at an effective Schrödinger equation¹² which describes the SMW

$$\text{d}_\xi^2 \psi + \left[Q^2 - \frac{1}{(1 + \xi^2)^2} \right] \psi = 0. \quad (27)$$

This ‘‘Schrödingerization’’ gives the possibility to use the quantum mechanical analogies¹³ and also the whole system of notions of quantum scattering theory. Constant Wronskians give also a technical convenience. As we will see later the amplification of SMW in 2D case ($k_z = 0$) is represented by the asymptotics phases of the wavefunctions Eq. (88); a simple illustration of Heisenberg 1938 idea that S-matrix contains whole experimentally accessible information.

As the effective potential $2mU/\hbar^2 = 1/(1 + \xi^2)^2$ is an even function of ξ , the general solution to the Schrödinger equation Eq. (27) is a linear combination of the even and odd solutions

$$\begin{aligned}\psi(\xi) &= C_g \psi_g(\xi) + C_u \psi_u(\xi), \\ \psi_g(-\xi) &= \psi_g(\xi), \quad \psi_u(-\xi) = -\psi_u(\xi),\end{aligned}\tag{28}$$

which obey the boundary conditions

$$\psi_g(0) = 1, \quad d_\xi \psi_g(0) = 0,\tag{29}$$

$$\psi_u(0) = 0, \quad d_\xi \psi_u(0) = 1.\tag{30}$$

The analytical solutions are represented by the confluent Heun function

$$\psi_g = \sqrt{1 + \xi^2} \text{HeunC}(0, -\frac{1}{2}, 0, -\frac{Q^2}{4}, \frac{1 + Q^2}{4}, -\xi^2),\tag{31}$$

$$\psi_u = \xi \sqrt{1 + \xi^2} \text{HeunC}(0, +\frac{1}{2}, 0, -\frac{Q^2}{4}, \frac{1 + Q^2}{4}, -\xi^2).\tag{32}$$

In the computer algebra system Maple the confluent Heun function reference is $\text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z)$. This special function obeys the equation

$$z(z - 1)y'' + [Az^2 + Bz + C]y' + [Dz + E]y = 0,\tag{33}$$

where

$$A = \alpha, \quad B = 2 + \beta + \gamma - \alpha, \quad C = -1 - \beta,\tag{34}$$

$$D = \frac{1}{2}[(2 + \gamma + \beta)\alpha + 2\delta], \quad E = \frac{1}{2}[-\alpha(1 + \beta) + (1 + \gamma)\beta + \gamma + 2\eta].\tag{35}$$

For the series expansion

$$y = \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} a_n z^n, \quad c_n \equiv a_n z^n\tag{36}$$

we arrive at the recursion

$$\mathbf{F}a_{n-1} + \mathbf{G}a_n + \mathbf{H}a_{n+1} = 0,\tag{37}$$

where

$$\mathbf{F} = \delta + \frac{1}{2}(2n + \beta\gamma),\tag{38}$$

$$\begin{aligned}\mathbf{G} &= n^2 + (1 + \gamma + \beta - \alpha)n + \frac{1}{2}[\gamma + 2\eta - \alpha + \beta(1 - \alpha + \gamma)] \\ \mathbf{H} &= -(n + 1)(n + 1 + \beta).\end{aligned}\tag{39}$$

Supposing $a_{-1} = 0$ and $a_0 = 1$ we use the recursion

$$a_{n+1} = -(\mathbf{F}a_{n-1} + \mathbf{G}a_n)/\mathbf{H}\tag{40}$$

which, for example, gives

$$a_1 = \frac{1}{2} [\beta(1 + \gamma - \alpha) + \gamma + 2\eta - \alpha] / (1 + \beta).\tag{41}$$

As the effective Schrödinger equation Eq. (27) has a solution for arbitrary $\xi \in (-\infty, +\infty)$, the formal series for Heun function have convergent Padé approximants.

Those Padé approximants can be calculated by the well-known ε -algorithm. First we calculate the series of the partial sums in zeroth (0) approximation

$$S_i^{(0)} = S_{i-1}^{(0)} + c_i \quad (42)$$

and for the first N terms we get

$$S_0^{(0)} = c_0, \quad S_1^{(0)} = c_0 + c_1, \quad \dots, \quad S_N^{(0)} = c_0 + c_1 + \dots + c_N. \quad (43)$$

Our problem is to calculate the limit of the sequence

$$S = \lim_{n \rightarrow \infty} S_n. \quad (44)$$

For this calculation we generate the auxiliary sequence

$$H_i^{(0)} = 1/c_{i+1} \quad (45)$$

recalling that $1/0 = 0$, i.e., using pseudoinverse numbers if we have to divide by zero:

$$H_0^{(0)} = 1/c_1, \quad H_1^{(0)} = 1/c_2, \quad \dots, \quad H_{N-1}^{(0)} = 1/c_N. \quad (46)$$

The epsilon-algorithm is the calculation of the recursion for the series

$$S_i^{(k)} = S_{i+1}^{(k-1)} + 1 / \left(H_{i+1}^{(k-1)} - H_i^{(k-1)} \right), \quad (47)$$

$$H_i^{(k)} = H_{i+1}^{(k-1)} + 1 / \left(S_{i+1}^{(k-1)} - S_i^{(k-1)} \right) \quad (48)$$

for all indices for which these relations make sense.

The maximal in modulus auxiliary element $H_{\max} = |H_{I+1}^{(K)}|$ gives the best Padé approximant $S_I^{(K)}$ for the searched limes and the accuracy is of the order of $1/H_{\max}$.

For each accuracy of the final result ϵ we can calculate S and H sequences with an accuracy δ to assure that $1/H_{\max} < \epsilon$. In such a way we obtain a method for calculating the confluent Heun function and the solution to the MHD equation in power series of time. If from physical arguments we know that a solution exists, the divergent series can be summed. Having a method for calculating the confluent Heun function, we can calculate both the even and odd solutions to Eq. (28). The accuracy in calculating Heun functions is controlled by the Wronskian

$$W(\psi_g, \psi_u) = \begin{vmatrix} \psi_g(\xi) & \psi_u(\xi) \\ d_\xi \psi_g(\xi) & d_\xi \psi_u(\xi) \end{vmatrix} = 1. \quad (49)$$

The constants from the general solution are also given by the Wronskians

$$C_g = W(\psi, \psi_u) = \psi(\xi_0) d_\xi \psi_u(\xi_0) - \psi_u(\xi_0) d_\xi \psi(\xi_0), \quad C_u = W(\psi_g, \psi) = \psi_g(\xi_0) d_\xi \psi(\xi_0) - \psi(\xi_0) d_\xi \psi_g(\xi_0). \quad (50)$$

Those formulae generally apply for a Cauchy problem where the initial conditions are imposed on the function being sought, i.e., on $\psi(\xi_0)$ and its derivative $d_\xi \psi_g(\xi_0)$.

All MHD variables can be expressed by the solution Eq. (28) to the effective Schrödinger equation. Then $b_x(\xi)$ is given by Eq. (26), and $b_z(\xi)$ by Eq. (22). Equation (25) gives b_y . And finally we know the time $\tau = Q\xi/K_y$ dependent amplitude of the magnetic field $\mathbf{b}(\xi = K_y\tau/Q)$. Analogously the velocity can be expressed by Eq. (12)

$$\mathbf{v}(\xi) = d_{Q\xi} \mathbf{b}(\xi), \quad d_{Q\xi} = \frac{d}{Q d\xi}, \quad (51)$$

i.e., plasma's displacement is parameterized by the magnetic field

$$\mathbf{b}(\xi) = Q \int \mathbf{v} d\xi = V_A k_y \int \mathbf{v} dt \quad (52)$$

recalling that magnetic field lines are frozen into the highly conductive fluid.

The physical time t is related to the dimensionless one τ

$$t = \frac{\tau}{A} = \frac{Q\xi}{AK_y}. \quad (53)$$

The time-dependence of the wave-vector Eq. (6) allows us to express the magnetic field $\mathbf{B}(t, \mathbf{r})$ and velocity of the fluid $\mathbf{V}(t, \mathbf{r})$ Eq. (7). The general solution depends on 4 arbitrary constants $C_g, C_u, \tilde{C}_g, \tilde{C}_u$.

After some algebra we can express also the wave component of the pressure¹⁵

$$p(t, \mathbf{r}) = \rho V_A^2 P(t) e^{i\mathbf{k}(t) \cdot \mathbf{r}}, \quad P(t) = -b_y(t) + 2 \frac{K_y v_x(t)}{K^2(t)}. \quad (54)$$

In such a way we derive a general analytical solution for linearized MHD waves in a shear flow of a magnetized fluid. In the next section, we will analyze the behavior of this exact solution when the waves amplification is significant.

The essential part of our analytical solution for three-dimensional SMWs in a shear flow is presented by the x -component of the magnetic field

$$\begin{aligned} B_x &= B_0 \operatorname{Re} \{ \exp [i (\mathbf{k}(t) \cdot \mathbf{r} + \varphi_0)] \} \\ &\times \left\{ C_g \operatorname{HeunC}(0, -\frac{1}{2}, 0, -\frac{Q^2}{4}, \frac{1+Q^2}{4}, -\xi^2) \right. \\ &\quad \left. + C_u \xi \operatorname{HeunC}(0, +\frac{1}{2}, 0, -\frac{Q^2}{4}, \frac{1+Q^2}{4}, -\xi^2) \right\}, \\ &= B_0 \cos [\mathbf{k}(t) \cdot \mathbf{r} + \varphi_0] \{ C_g b_{x,g}(\xi) + C_u b_{x,u}(\xi) \}, \end{aligned} \quad (55)$$

where

$$\xi = \frac{K_y}{Q} \tau = \frac{k_y}{\sqrt{k_y^2 + k_z^2}} A(t - t_0), \quad (56)$$

$$Q^2 = \Lambda^2 (k_y^2 + k_z^2), \quad \Lambda^2 = \frac{B_0^2}{\mu_0 \rho A^2}. \quad (57)$$

Here φ_0 and t_0 are arbitrary constants parameterizing the initial conditions. All other MHD variables: velocity, pressure and displacement of the plasma by the wave can be expressed by this solution. The dimensionless times have simple geometrical interpretation in the \mathbf{k} -space

$$\tau = -\frac{k_x(t)}{k_y}, \quad \xi = -\frac{k_x(t)}{\sqrt{k_y^2 + k_z^2}}. \quad (58)$$

For the velocity of AWs a substitution of ψ from Eq. (26) into Eq. (22) gives

$$v_z(\xi) = d_{Q\xi} b_z = -\tilde{C}_g \sin(Q\xi) + \tilde{C}_u \frac{\cos(Q\xi)}{Q} + \frac{2K_z}{Q} \int_{-\infty}^{\xi} \cos[Q(\xi - \xi')] \frac{v_x(\xi')}{1 + \xi'^2} d\xi'. \quad (59)$$

If we substitute here ψ from Eqs. (28), (31) and (32), we arrive at the final analytical solution which contains Heun functions.

III. AMPLIFICATION OF THE MHD WAVES

The odd and even solutions have asymptotics at $\xi \rightarrow \infty$

$$\psi_g \approx D_g \cos(Q\xi + \delta_g), \quad \psi_u \approx D_u \cos(Q\xi + \delta_u), \quad (60)$$

where the asymptotic phase shifts $\delta_g(Q^2)$, $\delta_u(Q^2)$ depend on the effective energy. For a sufficiently large wave-vectors we have the asymptotic

$$\delta_g(Q^2 \gg 1) \approx 0, \quad \delta_u(Q^2 \gg 1) \approx -\frac{\pi}{2}. \quad (61)$$

As we will see later, the averaged amplification coefficient of the energy of MHD waves $\mathcal{A}(\delta_g, \delta_u)$ depends only on the asymptotic phases of the solutions but not on the amplitudes D_g and D_u .

A. Auxiliary Quantum Mechanical Problem

Temporarily introducing imaginary exponents $\exp(i\mathbf{k} \cdot \mathbf{r})$ instead of $\sin(\mathbf{k} \cdot \mathbf{r})$ and $\cos(\mathbf{k} \cdot \mathbf{r})$ gives significant simplification of the analytical calculations related to the physics of the waves. In this subsection we will consider ψ in Eq. (27) as a complex function in order to make easier any further analysis of the MHD amplification coefficient. In order to analyze the effective MHD equation Eq. (27) we will solve the quantum-mechanical counterpart of our MHD problem, that is a tunneling through a barrier $2mU/\hbar^2 = 1/(1 + \xi^2)^2$, supposing that ψ is a complex function. Consider an incident wave with a unit amplitude, a reflected wave with amplitude R and a transmitted wave with amplitude T

$$\psi(\xi \rightarrow -\infty) \approx \exp(iQ\xi) + R \exp(-iQ\xi), \quad (62)$$

$$\psi(\xi \rightarrow +\infty) \approx T \exp(+iQ\xi). \quad (63)$$

Using the asymptotics of the eigen-functions

$$\psi_g \approx \begin{cases} D_g \cos(Q\xi - \delta_g) & \text{for } \xi \rightarrow -\infty, \\ D_g \cos(Q\xi + \delta_g) & \text{for } \xi \rightarrow +\infty, \end{cases} \quad (64)$$

and

$$\psi_u \approx \begin{cases} -D_u \cos(Q\xi - \delta_u) & \text{for } \xi \rightarrow -\infty, \\ D_u \cos(Q\xi + \delta_u) & \text{for } \xi \rightarrow +\infty \end{cases} \quad (65)$$

as well as the general condition

$$\psi(\xi) = C_g^{(q)} \psi_g(\xi) + C_u^{(q)} \psi_u(\xi), \quad (66)$$

we compare the coefficients in front of $\exp(iQ\xi)$ and $\exp(-iQ\xi)$ for $\xi \rightarrow -\infty$ and $\xi \rightarrow \infty$. The solution to a simple matrix problem yields

$$C_g^{(q)} D_g = \exp(i\delta_g), \quad C_u^{(q)} D_u = -\exp(i\delta_u) \quad (67)$$

Then for the tunneling amplitude we get

$$T = |T| e^{i(\delta_u + \delta_g - \pi/2)} \quad (68)$$

and finally for the tunneling coefficient we obtain

$$\mathcal{D} = |T|^2 = s_{ug}^2, \quad s_{ug} \equiv \sin(\delta_u - \delta_g). \quad (69)$$

The convenience of the tunneling coefficient is that it varies in the range $0 \leq \mathcal{D} \leq 1$. In the next two subsections we will present the SMW amplification coefficient as a function of the tunneling coefficient $\mathcal{K} = 2/\mathcal{D} - 1$ on the analogy of Heisenberg's ideas in quantum mechanics where the statistical properties of the scattering problem depend only on the phases of the S -matrix.

The phases $\delta_g, \delta_u \in (-\pi, \pi)$ and amplitudes D_g and D_u can be determined continuing the exact wave functions Eqs. (31)–(32) with WKB asymptotics Eq. (60). Thus we obtain

$$\begin{aligned} \psi &= D \cos(Q\xi + \delta), \quad \tilde{\delta} = -Q\xi - \arctan \frac{d_\xi \psi(\xi)}{Q\psi(\xi)}, \\ \delta &= \tilde{\delta} - 2\pi \times \text{int} \left(\frac{\tilde{\delta} + \pi}{2\pi} \right) \in (-\pi, \pi), \quad D = \frac{\psi(\xi)}{\cos(Q\xi + \delta)}, \end{aligned} \quad (70)$$

at some sufficiently large $\xi \gg 1 + 2\pi/Q$. Here $\text{int}(\dots)$ stands for the integer part of a real number. When programming we have to use the two-argument arctan function

$$\arctan(y, x) = \arctan(y/x) + \frac{\pi}{2} \theta(-x) \text{sgn}(y) \in (-\pi, \pi). \quad (71)$$

The accuracy of this continuation is controlled by the Wronskian from the asymptotic wave functions

$$W(\psi_g, \psi_u)(\xi) = Q D_g D_u \sin(\delta_g - \delta_u) = 1. \quad (72)$$

B. MHD and Real ψ

Imagine that in an ideal plasma we have at $t \rightarrow -\infty$ some plane MHD wave – our task is to calculate how many times the energy density increases at $t \rightarrow \infty$, and to average this amplification over all the initial phases of that wave. As there is no amplification for the b_z component according to Eq. (16) we will concentrate our attention on the b_x component. The amplification comes from the negative “friction” term $\propto \xi/(1+\xi^2)$ in Eq. (15). The influence of this friction is transmitted to the effective potential barrier $\propto 1/(1+\xi^2)$. For $Q^2 < 1$ we have an analog of the quantum mechanical tunneling.

In the current MHD problem ψ is a real variable with asymptotics

$$\psi \approx \begin{cases} \cos(Q\xi - \phi_i) & \text{for } \xi \rightarrow -\infty, \\ D_f \cos(Q\xi + \phi_f) & \text{for } \xi \rightarrow +\infty, \end{cases} \quad (73)$$

i.e., we have an incident wave with a unit amplitude and an initial phase ϕ_i . D_f is the amplitude and ϕ_f is the phase of the amplified wave.

Again we present the ψ function as linear combination of even and odd solutions

$$\psi(\xi) = C_g^{(c)} \psi_g(\xi) + C_u^{(c)} \psi_u(\xi). \quad (74)$$

Here we substitute the asymptotic formulas Eq. (64) and Eq. (65), and the comparison of the coefficient with Eq. (73) at $\xi \rightarrow -\infty$ gives

$$C_g^{(c)} D_g = \frac{s_{iu}}{s_{gu}}, \quad C_u^{(c)} D_u = \frac{s_{ig}}{s_{gu}}, \quad (75)$$

where

$$s_{iu} \equiv \sin(\phi_i - \delta_u), \quad s_{ig} \equiv \sin(\phi_i - \delta_g). \quad (76)$$

The comparison of the coefficients at $\xi \rightarrow \infty$ gives for the phase and the amplification of the signal

$$\phi_f = F(\phi_i) \equiv \arctan \frac{s_{ig}s_u + s_{iu}s_g}{s_{ig}c_u + s_{iu}c_g}, \quad (77)$$

$$\mathcal{A}(\phi_i) \equiv D_f^2 = \frac{\mathcal{N}}{\mathcal{D}}, \quad (78)$$

where

$$\mathcal{N} = (s_{ig}s_u + s_{iu}s_g)^2 + (s_{ig}c_u + s_{iu}c_g)^2, \quad (79)$$

$$s_g = \sin(\delta_g), \quad s_u = \sin(\delta_u), \quad (80)$$

$$c_g = \cos(\delta_g), \quad c_u = \cos(\delta_u). \quad (81)$$

The reversibility of the dissipation-free motion leads us to $\phi_i = F(\phi_f)$, i.e., function F coincides with its inverse function $F(F(\phi)) = \phi$. As time reverses the wave amplification is converted to attenuation (damping in some sense) $\mathcal{A}(\phi_i)\mathcal{A}(F(\phi_i)) = 1$.

In unabridged mathematical notations we have the function

$$F(\varphi) \equiv \arctan \frac{\sin(\varphi - \alpha) \sin \beta + \sin(\varphi - \beta) \sin \alpha}{\sin(\varphi - \alpha) \cos \beta + \sin(\varphi - \beta) \cos \alpha}, \quad (82)$$

defined in the interval $\varphi \in (-\pi/2, \pi/2)$. For arbitrary values of the parameters α and β

$$F[F(\phi)] = \phi, \quad (83)$$

i.e., this function F coincides with its inverse function F^{-1} . The nonlinear function F has only 2 immovable points

$$F(\alpha) = \alpha, \quad F(\beta) = \beta. \quad (84)$$

Defining also

$$\mathcal{A}(\varphi) \equiv \left\{ [\sin(\varphi - \alpha) \sin \beta + \sin(\varphi - \beta) \sin \alpha]^2 + [\sin(\varphi - \alpha) \cos \beta + \sin(\varphi - \beta) \cos \alpha]^2 \right\} / \sin^2(\alpha - \beta), \quad (85)$$

we have another curious relation

$$\mathcal{A}[F(\varphi)] \mathcal{A}(\varphi) = 1. \quad (86)$$

The so derived amplification coefficient $\mathcal{A}(\phi_i; \delta_g, \delta_u)$ depends on the initial phase. In the next subsection we will consider the statistical problem of phase averaging with respect to the initial phase ϕ_i .

C. Phase Averaged Amplification

For waves generated by turbulence the initial phase is unknown and one can suppose a uniform phase distribution. That is why for solving the statistical problem of energy amplification we need to calculate average values with respect to the initial phase ϕ_i . That idea is coming from the well-known random phase approximation (RPA) in plasma physics. The phase averaging already introduces an element of irreversibility because we already suppose that waves are created with random phases. This is the MHD analog of the molecular chaos from the theorem for entropy increase in the framework of the kinetic theory if the probability distributions are introduced in the initial conditions of the mechanical problem. In the case of accretion flows we also suppose that turbulence is a chaotic phenomenon and we have to apply the RPA for investigating the statistical properties.

The calculation of the integral

$$\int_0^\pi \frac{d\phi_i}{\pi} \mathcal{N}(\phi_i) = 2 - s_{ug}^2 = 2 - \mathcal{D} \quad (87)$$

gives for the initial phase averaged gain

$$\mathcal{G} \equiv \int_{-\pi/2}^{\pi/2} \mathcal{A}(\phi_i) \frac{d\phi_i}{\pi} = \frac{2}{\sin^2(\delta_u - \delta_g)} = \frac{2}{\mathcal{D}} - 1. \quad (88)$$

In such a way the SMW amplification coefficient \mathcal{G} is presented by the tunneling coefficient \mathcal{D} of the corresponding quantum problem. Both coefficients are expressed by the asymptotic phases δ_g , δ_u in analogy with partial waves phase analysis of the quantum mechanical scattering problem in atomic and nuclear physics. The axial symmetry of this result $\mathcal{G} \left(\sqrt{K_y^2 + K_z^2} \right)$ significantly simplifies the further statistical analysis.

Let us analyze the physical meaning of the gain coefficient \mathcal{G} . As

$$Q\xi = K_y\tau = V_A k_y t = \omega_A \text{sgn}(k_y)t, \quad \omega_A(\mathbf{k}) = |\mathbf{V}_A \cdot \mathbf{k}| = V_A |k_y| \geq 0, \quad (89)$$

$$\mathbf{v}_{gr} \equiv \frac{\partial \omega_A}{\partial \mathbf{k}} = \mathbf{V}_A \text{sgn}(\mathbf{V}_A \cdot \mathbf{k}) = V_A \text{sgn}(k_y) \mathbf{e}_y, \quad (90)$$

the asymptotics Eq. (73) let us conclude that for $|t| \rightarrow \infty$ we have only magnetosonic waves with dispersion coinciding with that of Alfvén waves.¹⁶ In the spirit of M. T. Weiss quantum interpretation of the classical Manley–Rowe theorem^{17,18} one can present the wave energy $\hbar\omega_A N$ by a number of quanta, the number of *alfvenons*: “The alfvénons introduced in this Letter¹⁹ appear to be effective and spectacular converters of electromagnetic energy flux into kinetic energy of particles.” We use this notion in a slightly different sense, our former terminology was *alfvons*¹⁵ in our case. Following this interpretation, the energy gain \mathcal{G} describes the increasing number of quanta

$$\mathcal{G} \left(\sqrt{k_y^2 + k_z^2} \right) = \frac{\hbar\omega_A(|k_y|)\overline{N}_{\text{alfvenons}}(t \rightarrow +\infty)}{\hbar\omega_A(|k_y|)\overline{N}_{\text{alfvenons}}(t \rightarrow -\infty)}, \quad (91)$$

as in a laser system. In this terminology the mechanism of heating of quasars can be phrased, namely due to lasing of alfvénons in shear flows of magnetized plasma. Laser or rather maser²¹ effects are typical phenomena in space plasmas. The hydrodynamic overreflection instability² and burst-like increase of the wave amplitude¹² are phenomena of similar kind. More precisely \mathcal{G} is the gain for the x - y -polarized SMWs, the energy of mode conversion in z -polarized AWs will be analyzed elsewhere. The notion amplification is correct for standing waves but $2\cos(\mathbf{k} \cdot \mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} + e^{-i\mathbf{k} \cdot \mathbf{r}}$ and amplification is simultaneous for opposite wave-vectors. That is why some theorists prefer to use overreflection in spite that there is no rigid object reflecting the waves.

D. Analytical Approximations for Amplification

For scattering problems by a localized potential at small wave-vectors $Q \ll 1$, when the wavelength is much larger than the typical size of the nonzero potential, we can apply the delta-function approximation

$$\frac{2m}{\hbar^2} U(\xi) \equiv \frac{1}{(1 + \xi^2)^2} \rightarrow 2Q_0 \delta(\xi). \quad (92)$$

In this well-known quantum mechanical problem²² the transmission coefficient is

$$D \approx \frac{Q^2}{Q^2 + Q_0^2}, \quad \delta_u = -\frac{\pi}{2}, \quad \delta_g = \frac{\pi}{2} + \arctan \frac{Q}{Q_0}, \quad \psi_g = \frac{\cos(|Q\xi| + \delta_g)}{\cos(\delta_g)}, \quad \psi_u = \frac{\sin Q\xi}{Q}. \quad (93)$$

According to the tradition of the method of potential of zero radius, the coefficient $Q_0 \approx \frac{1}{2}\pi$ is determined by the behavior of the phases at small wave-vectors. Only qualitatively this parameter corresponds to the area of the potential

$$2Q_0 = \frac{\pi}{2}Z_{\text{ren}} = \pi, \quad \frac{\pi}{2} = \int_{-\infty}^{\infty} \frac{1}{(1+\xi^2)^2} d\xi, \quad (94)$$

but the renormalizing coefficient $Z_{\text{ren}} = 2$ which we have to introduce differs from unity. According to the general relation Eq. (88) for the amplification we have

$$\mathcal{G} - 1 = 2 \left(\frac{1}{\mathcal{D}} - 1 \right) \approx \frac{2Q_0^2}{Q^2}, \quad 2Q_0^2 = \frac{\pi^2}{2} \approx 4.934 \approx 5. \quad (95)$$

The scattering phases δ_g and δ_u can be obtained by a fit of the asymptotic wave functions Eq. (64) and Eq. (65) with the analytical solutions Eq. (31) and Eq. (32). In such a way we can calculate the wave-vector dependence of the transmission coefficient as it is depicted in Fig. 1.

Let us derive the analytical formula for Q_0 : The solutions

$$\psi_g(\xi) = \sqrt{1+\xi^2}, \quad (96)$$

$$\psi_u(\xi) = \sqrt{1+\xi^2} \arctan(\xi) \quad (97)$$

of effective Schrödinger's equation Eq. (27) in long wavelength limit $Q \rightarrow 0$

$$d_\xi^2 \psi - \frac{1}{(1+\xi^2)^2} \psi = 0 \quad (98)$$

with asymptotics at $\xi \rightarrow \infty$

$$\psi_g(\xi \gg 1) \approx \xi, \quad (99)$$

$$\psi_u(\xi \gg 1) \approx \frac{\pi}{2} \xi \quad (100)$$

have to be compared with the approximative solutions Eq. (64) and Eq. (65) for $\xi \gg 1$, when for $Q \rightarrow 0$ we have $|\delta_g| \approx |\delta_u| \approx \frac{\pi}{2}$ and sinusoids are almost linear

$$\psi_g(1 \ll \xi \ll \frac{1}{Q}) \approx D_g \sin(Q\xi) \approx D_g Q\xi, \quad (101)$$

$$\psi_u(1 \ll \xi \ll \frac{1}{Q}) \approx D_u \sin(Q\xi) \approx D_u Q\xi. \quad (102)$$

The comparison of the first derivatives in this region

$$d_\xi \psi_g(\xi \gg 1) \approx 1 \approx D_g Q, \quad (103)$$

$$d_\xi \psi_u(\xi \gg 1) \approx \frac{\pi}{2} \approx D_u Q \quad (104)$$

gives

$$D_g \approx \frac{1}{Q}, \quad D_u \approx \frac{\pi}{2Q}. \quad (105)$$

Then formula for Wronskian Eq. (72) determines the phase difference

$$\sin(\delta_g - \delta_u) \approx \frac{2}{\pi} Q \quad (106)$$

and transmission coefficient

$$\mathcal{D} \approx \sin^2(\delta_g - \delta_u) = \left(\frac{2Q}{\pi} \right)^2. \quad (107)$$

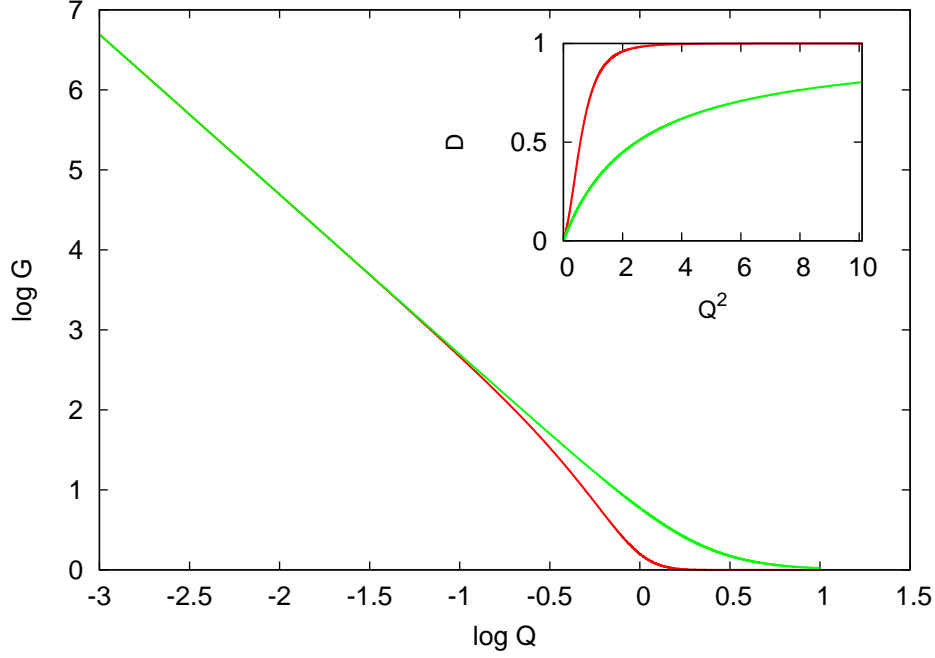


FIG. 1: The efficiency of the dissipation mechanism that is likely responsible for the formation of stars and other compact astrophysical objects: logarithm of the wave energy amplification $\mathcal{G} = 2/\mathcal{D} - 1$ (cf. with Eq. (88)) versus the logarithm of the dimensionless wave-vector Q . The delta-function approximation Eq. (95) (upper green curve) is asymptotically exact at huge (giant) amplifications. In the inset the transmission coefficient \mathcal{D} is plotted versus the square of the wave-vector Q^2 of the auxiliary quantum mechanical problem. The delta-function potential approximation (lower green curve) Eq. (93) is good only for long enough wavelengths $Q^2 \ll 1$ at which the amplification is significant $\mathcal{G} \gg 1$; $\log \mathcal{G} = 6$ means energy amplification one million times or 60 dB.

The Eq. (88) then gives for the phase averaged amplification

$$G = \frac{2}{D} - 1 \approx \frac{\pi^2}{2Q^2} + \text{const} \quad (108)$$

and the comparison with δ -potential approach Eq. (95) gives the analytical result for the strength of the δ -potential $Q_0 = \pi/2$ given in Eq. (95).

Our problem formally coincides with the quantum problem¹³ of transmission coefficient $\mathcal{D} \propto E$ at low energies

$$U(x) = \frac{U_0}{[1 + (\alpha x)^2]^2}, \quad U_0 = \frac{\hbar^2 \alpha^2}{2m}, \quad E = U_0 Q^2, \quad D \approx \left(\frac{2Q}{\pi} \right)^2 = \frac{4}{\pi^2} \frac{E}{U_0} \ll 1. \quad (109)$$

The zero-radius potential Padé approximant Eq. (93) has an acceptable accuracy for small wave-vectors. Having the quantum mechanical transmission coefficient, we can calculate the energy gain coefficient $\mathcal{G} - 1$. In the next section we will incorporate the approximative solution

$$\mathcal{G} - 1 \sim \frac{1}{Q^2} = \frac{A^2}{V_A^2 q^2}, \quad q^2 = k_\perp^2 \equiv k_y^2 + k_z^2 = \text{const} \quad (110)$$

in the simplest model of Kolmogorov turbulence. Here the subscript \perp means that the amplification depends on the projection of the wave-vector perpendicular to the shear velocity and magnetic field.

The delta function approximation has a visual interpretation in classical mechanics as well, supposing that ψ is the displacement of an oscillator and ξ is the time. The approximative equation

$$d_\xi^2 \psi = -Q^2 \psi(\xi) + 2Q_0 \delta(\xi) \psi(\xi) \quad (111)$$

means that at a time moment $\xi = 0$ the oscillator is subjected to a forcing impulse with a magnitude

$$d_\xi \psi(+0) - d_\xi \psi(-0) = 2Q_0 \psi(0), \quad \text{or} \quad v_x(+0) - v_x(-0) = \frac{\pi b_x(0)}{\sqrt{K_y^2 + K_z^2}}, \quad \text{when} \quad K_x(0) = 0. \quad (112)$$

If for $Q \ll Q_0$ we have initial oscillations with amplitude A_i

$$\psi = A_i \cos Q\xi \quad \text{for } \xi < 0, \quad (113)$$

after the push in $\xi = 0$ we have oscillations with much increased amplitude

$$\psi = A_f \sin Q\xi, \quad \text{for } \xi > 0, \quad A_f \approx \frac{2Q_0}{Q} A_i \gg A_i. \quad (114)$$

The strong push with an appropriate phase “amplifies” the oscillations; this burst-like increase of the wave amplitude was observed in numerical investigations of linearized two-dimensional MHD equations.^{11,12} This phenomenon is akin to the extremely strong hydrodynamic instabilities due to a velocity jump; its prediction and discovery both in theory and experiments are described in Ref. [23]. In such a way the Alfvén’s idea of the importance of MHD waves in the transfer of momentum is reduced to the very simple mathematics for the jump of the velocity of SMWs Eq. (112). It is instructive to rewrite this force in the \mathbf{r} -space.

IV. INCORPORATION OF TURBULENCE AS RANDOM DRIVER OF MHD WAVES

Being as efficient as has been demonstrated above, the amplification of SMWs, Eq. (110), is possibly the dominant physical factor responsible for generating and maintaining the turbulence in accretion flows. However the theory of turbulence is much more complicated than the theory of linearized waves. That is why here we provide an illustration how the wave amplification can be incorporated in the turbulence theory. In order to establish common set of notions and notations we will recall some basic properties of the homogeneous isotropic Kolmogorov–Obukhov turbulence.

A. Kolmogorov Turbulence

Let the velocity be presented by the Fourier integral

$$\mathbf{V}(\mathbf{r}) = \int e^{i\mathbf{k}\cdot\mathbf{r}} \mathbf{V}_{\mathbf{k}} \frac{d^3k}{(2\pi)^3}, \quad \mathbf{V}_{\mathbf{k}} = \int \mathbf{V}(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} d^3x. \quad (115)$$

The energy per unit mass is

$$\int \frac{1}{2} V^2(\mathbf{r}) d^3x = \int \frac{1}{2} V_{\mathbf{k}}^2 \frac{d^3k}{(2\pi)^3} = \int \mathcal{E}_{\mathbf{k}} \frac{d^3k}{(2\pi)^3}, \quad (116)$$

where we introduce the spectral density averaged with respect to the turbulence

$$\mathcal{E}_{\mathbf{k}} = \frac{1}{2} \langle V_{\mathbf{k}}^2 \rangle_{\text{turbulence}}. \quad (117)$$

We can introduce also the energy of vortices which contains Fourier components with wavelength $2\pi/k$, shorter than some fixed length λ

$$\frac{1}{2} V_{\lambda}^2 = \int_{k < 1/\lambda} \mathcal{E}_{\mathbf{k}} \frac{d^3k}{(2\pi)^3}, \quad (118)$$

where for isotropic turbulence $\mathcal{E}_{\mathbf{k}} = \mathcal{E}_k$. This energy evaluates turbulent pulsation with size λ . Further on we will continue with only order of magnitude evaluations, hence in the following estimations we will drop off factors such as 4π , $\frac{1}{2}$, etc.

According to the Kolmogorov–Obukhov (KO) scenario in the inertial range the magnitude of the velocity pulsations V_{λ} can depend only on the turbulent power dissipated per unit mass ε . There is only one combination with the appropriate dimension

$$\varepsilon = \frac{V_{\lambda}^2}{\lambda/V_{\lambda}} = \frac{\text{energy/mass}}{\text{time} = \text{length/velocity}} = \frac{\text{power}}{\text{mass}}, \quad (119)$$

which yields

$$\begin{aligned} V_\lambda^2 &\sim (\varepsilon\lambda)^{2/3} \sim \int_{k\lambda>1} \mathcal{E}_\mathbf{k}^{\text{KO}} d^3k \sim \int_{k\lambda>1} \frac{\varepsilon^{2/3}}{k^{5/3}} dk, & d^3k &\sim k^2 dk, \\ E(k) &= \int k^2 \mathcal{E}_\mathbf{k}^{\text{KO}} d\Omega \approx C_K \varepsilon^{2/3} k^{-5/3}, & C_K &\approx 1.6, & \mathcal{E}_\mathbf{k}^{\text{KO}} &\sim \frac{\varepsilon^{2/3}}{k^{11/3}}. \end{aligned} \quad (120)$$

V_λ is the amplitude of variation in the velocity pulsation at distance λ . $\mathcal{E}_\mathbf{k}$ is the energy density in the \mathbf{k} -space per unit mass; in the Kolmogorov–Obukhov picture this is a static variable.

The scaling law $V_\lambda \sim (\varepsilon\lambda)^{1/3}$ Eq. (119) is applicable for large enough distances $\lambda > \lambda_0$, where λ_0 describes the scale where dissipation effects become essential

$$\begin{aligned} \lambda_0 &\equiv (\nu_k^3/\varepsilon)^{1/4}, & \nu_k &= \eta/\rho, & \rho &= MN_p, & N_p &= N_e, & \eta &= 0.4 \frac{M^{1/2} T_p^{5/2}}{e^4 \mathcal{L}_p}, \\ \mathcal{L}_p &= \ln \left(\frac{\lambda_D T_p}{e^2} \right), & \frac{1}{\lambda_D^2} &= 4\pi e^2 \left(\frac{N_e}{T_e} + \frac{N_p}{T_p} \right), & e^2 &= \frac{q_e^2}{4\pi\varepsilon_0}, & \mathcal{L}_e &= \ln \left(\frac{\lambda_D T_e}{e^2} \right), \\ p &= N_p T_p + N_e T_e, & \frac{M}{m} &\approx 1836, & c_s^2 &= \frac{5}{3} \frac{T_p + T_e}{M} = \frac{5p}{3\rho}, & \nu_k &= \frac{\eta}{\rho} = \frac{0.4 T_p^{5/2}}{e^4 N_p M^{1/2} \mathcal{L}_p}, \\ \kappa &= \frac{0.9 T_e^{5/2}}{e^4 m^{1/2} \mathcal{L}_e}, & C_V &= \frac{3}{2} (N_e + N_p), & \chi &= \frac{\kappa}{C_V}, & P &= \frac{\nu_k}{\chi} = 1.3 \frac{\mathcal{L}_e}{\mathcal{L}_p} \left(\frac{m}{M} \right)^{1/2} \left(\frac{T_p}{T_e} \right)^{5/2} \ll 1, \\ \nu_m &= \varepsilon_0 c^2 \tilde{\varrho} = \frac{e^2 c^2 m^{1/2} \mathcal{L}_e}{0.6 \times 4\pi T_e^{3/2}}, & \tilde{\varrho} &= \frac{1}{4\pi\varepsilon_0} \frac{e^2 m^{1/2} \mathcal{L}_e}{0.6 T_e^{3/2}}, & P_m &= \frac{\nu_k}{\nu_m} = \frac{0.24 \times 4\pi T_e^{3/2} T_p^{5/2}}{mc^2 e^6 N_p} \gg 1. \end{aligned} \quad (121)$$

Here, apropos, we introduced self-explanatory notations for the parameters of fully ionized hydrogen plasma: bare plasma viscosity η , the Coulomb logarithm for protons \mathcal{L}_p , the proton mass M , mass density $\rho = MN_p$, pressure p , sound speed c_s , magnetic diffusivity ν_m , kinematic viscosity ν_k , heat conductivity κ , specific heat per unit volume C_v , temperature conductivity χ , Prandtl number P , magnetic Prandtl number P_m , total viscosity determining Alfvén waves damping $\nu = \nu_k + \nu_m$, Ohmic resistivity $\tilde{\varrho}$, constant in Coulomb interaction e^2 , Debye screening length λ_D , electron temperature T_e and proton temperature T_p times the Boltzmann constant, number of electrons and protons per unit volume $N_e = N_p$, etc. Those formulas are system invariant: in SI $\mu_0 = 4\pi \times 10^{-7} = 1/c^2\varepsilon_0$, in Gaussian system $\mu_0 = 4\pi = 1/\varepsilon_0$, while in the Heaviside–Lorentz system $\mu_0 = 1 = 1/\varepsilon_0$.

Let us now consider a magnetosonic wave with a time-dependent wave-vector Eq. (6)

$$q_x(t) = -A(t - t_0)q_y, \quad q_y = \text{const}, \quad q_z = \text{const}, \quad (122)$$

and time-dependent energy density per unit mass in real space

$$w(t) = \frac{1}{2} \langle \mathbf{V}_{\text{wave}}^2 + \frac{\mathbf{B}_{\text{wave}}^2}{\rho\mu_0} \rangle = \frac{V_A^2}{4} \left[\mathbf{b}^2 + (d_{Q\xi} \mathbf{b})^2 \right], \quad (123)$$

where $\langle \dots \rangle$ stands for spatial averaging, $\langle \cos^2(\mathbf{k} \cdot \mathbf{r}) \rangle = \frac{1}{2}$. Then the energy density in the \mathbf{k} -space is

$$\mathcal{E}_\mathbf{k}(t) = w(t) \delta[\mathbf{k} - \mathbf{q}(t)]. \quad (124)$$

Let us mention that all MHD variables \mathbf{b} , \mathbf{v} , w in Eq. (123), and P in Eq. (54) depend on the effective wave functions

ψ and χ (solutions to the effective Schrödinger equations) through the dimensionless time ξ :

$$b_x = \frac{\psi(\xi)}{\sqrt{1+\xi^2}}, \quad (125)$$

$$b_y = -\frac{2K_z^2}{K_y Q} \int_{-\infty}^{\xi} \sin[Q(\xi - \xi')] \frac{v_x(\xi')}{1+\xi'^2} d\xi' + \frac{Q}{K_y} \frac{\xi\psi(\xi)}{\sqrt{1+\xi^2}} - \frac{K_z}{K_y} \chi(\xi), \quad (126)$$

$$b_z = \frac{2K_z}{Q} \int_{-\infty}^{\xi} \sin[Q(\xi - \xi')] \frac{v_x(\xi')}{1+\xi'^2} d\xi' + \chi(\xi), \quad (127)$$

$$v_x = \frac{(1+\xi^2)d_\xi\psi(\xi) - \xi\psi(\xi)}{Q(1+\xi^2)^{3/2}}, \quad (128)$$

$$v_y = -\frac{2K_z^2}{K_y Q} \int_{-\infty}^{\xi} \cos[Q(\xi - \xi')] \frac{v_x(\xi')}{1+\xi'^2} d\xi' + \frac{\xi(1+\xi^2)d_\xi\psi(\xi) - \xi^2\psi(\xi)}{K_y(1+\xi^2)^{3/2}} - \frac{K_z}{QK_y} d_\xi\chi(\xi), \quad (129)$$

$$v_z = \frac{2K_z}{Q} \int_{-\infty}^{\xi} \cos[Q(\xi - \xi')] \frac{v_x(\xi')}{1+\xi'^2} d\xi' + \frac{d_\xi\chi}{Q}. \quad (130)$$

For programming one can use also formulae explicitly expressed by the initial conditions $\mathbf{b}(t_0)$, $\mathbf{v}(t_0)$

$$b_x = \frac{\psi(\xi)}{\sqrt{1+\xi^2}}, \quad \xi = -\frac{K_x(\tau)}{\sqrt{K_y^2 + K_x^2}} = -\frac{K_x(\tau_0)}{Q} + \frac{K_y}{Q}(\tau - \tau_0), \quad K_x(\tau) = K_x(\tau_0) - K_y(\tau - \tau_0), \quad (131)$$

$$b_y = -\frac{2K_z^2}{QK_y} [\sin(Q\xi)I_c(\xi) - \cos(Q\xi)I_s(\xi)] + \frac{Q}{K_y} \frac{\xi\psi(\xi)}{\sqrt{1+\xi^2}} - \frac{K_z}{K_y} \chi(\xi) = \frac{Q}{K_y} \xi b_x - \frac{K_z}{K_y} b_z, \quad (132)$$

$$b_z = \frac{2K_z}{Q} [\sin(Q\xi)I_c(\xi) - \cos(Q\xi)I_s(\xi)] + \chi(\xi), \quad (133)$$

$$v_x = \frac{(1+\xi^2)d_\xi\psi(\xi) - \xi\psi(\xi)}{Q(1+\xi^2)^{3/2}} = d_{Q\xi}b_x, \quad (134)$$

$$\begin{aligned} v_y &= -\frac{2K_z^2}{QK_y} [\cos(Q\xi)I_c(\xi) + \sin(Q\xi)I_s(\xi)] + \frac{\xi(1+\xi^2)d_\xi\psi(\xi) - \xi^2\psi(\xi)}{K_y(1+\xi^2)^{3/2}} - \frac{K_z}{K_y} d_{Q\xi}\chi(\xi) \\ &= \frac{Q}{K_y} \xi v_x - \frac{K_z}{K_y} v_z, \end{aligned} \quad (135)$$

$$v_z = \frac{2K_z}{Q} [\cos(Q\xi)I_c(\xi) + \sin(Q\xi)I_s(\xi)] + d_{Q\xi}\chi, \quad (136)$$

where

$$I_c(\xi) \equiv \int_{\xi_0}^{\xi} \cos(Q\xi') \frac{v_x(\xi')}{1+\xi'^2} d\xi' = \int_{\xi_0}^{\xi} \cos(Q\xi') \frac{(1+\xi'^2)d_{\xi'}\psi(\xi') - \xi'\psi(\xi')}{Q(1+\xi'^2)^{5/2}} d\xi', \quad (137)$$

$$I_s(\xi) \equiv \int_{\xi_0}^{\xi} \sin(Q\xi') \frac{v_x(\xi')}{1+\xi'^2} d\xi' = \int_{\xi_0}^{\xi} \sin(Q\xi') \frac{(1+\xi'^2)d_{\xi'}\psi(\xi') - \xi'\psi(\xi')}{Q(1+\xi'^2)^{5/2}} d\xi', \quad (138)$$

$$\chi(\xi) = b_z(\xi_0) \cos[Q(\xi - \xi_0)] + v_z \sin[Q(\xi - \xi_0)], \quad \xi_0 = -\frac{k_x(t_0)}{\sqrt{k_y^2 + k_z^2}} = -\frac{K_x(\xi_0)}{Q}, \quad (139)$$

$$d_{Q\xi}\chi(\xi) = -b_z(\xi_0) \sin[Q(\xi - \xi_0)] + v_z \cos[Q(\xi - \xi_0)], \quad K_x(\xi) = -Q\xi = K_x(\xi_0) + Q(\xi - \xi_0), \quad (140)$$

$$\psi(\xi_0) = \sqrt{1+\xi_0^2} b_x(\xi_0), \quad d_\xi\psi(\xi_0) = Q\sqrt{1+\xi_0^2} v_x(\xi_0) + \frac{\xi_0}{\sqrt{1+\xi_0^2}} b_x(\xi_0), \quad (141)$$

$$\psi(\xi) = [\psi(\xi_0)d_\xi\psi_u(\xi_0) - \psi_u(\xi_0)d_\xi\psi(\xi_0)] \psi_g(\xi) + [\psi_g(\xi_0)d_\xi\psi(\xi_0) - \psi(\xi_0)d_\xi\psi_g(\xi_0)] \psi_u(\xi). \quad (142)$$

In longwavelength limit $Q \ll 1$ and $\xi_0 \rightarrow -\infty$ the numerical integration gives

$$J_{s,g} \equiv \int_{-\infty}^{\infty} \sin(Q\xi') \frac{(1+\xi'^2)d_{\xi'}\psi_g(\xi') - \xi'\psi_g(\xi')}{Q(1+\xi'^2)^{5/2}} d\xi' \approx -0.302 \sim 1, \quad (143)$$

$$J_{c,u} \equiv \int_{-\infty}^{\infty} \cos(Q\xi') \frac{(1+\xi'^2)d_{\xi'}\psi_u(\xi') - \xi'\psi_u(\xi')}{Q(1+\xi'^2)^{5/2}} d\xi' \approx \frac{C_{c,u}}{Q} \gg |J_{s,g}|, \quad C_{c,u} \approx 1.459 \sim 1. \quad (144)$$

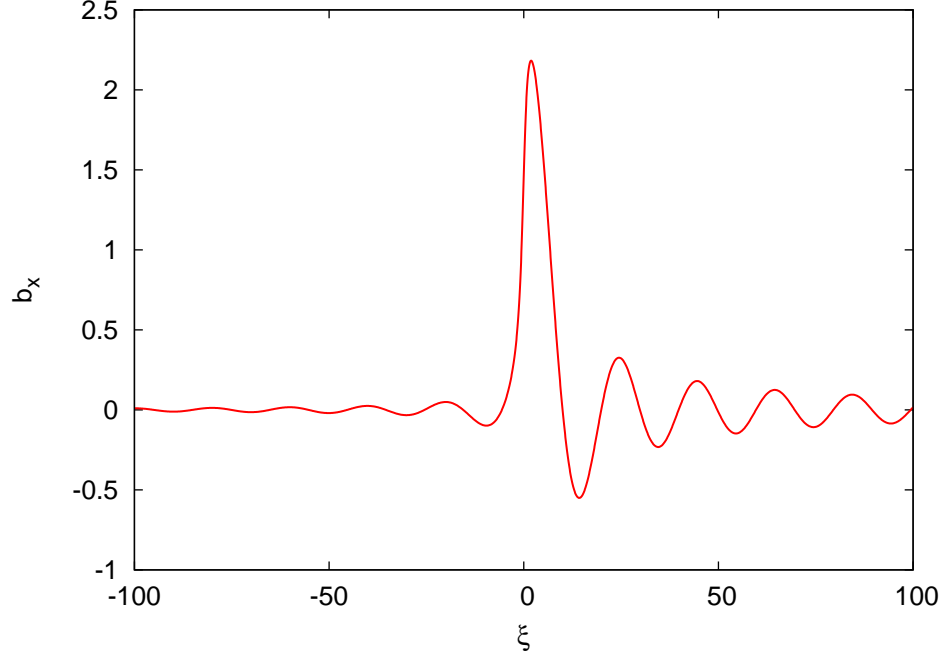


FIG. 2: Radial wave component of the magnetic field $b_x(\xi)$ as function of the dimensionless time ξ .

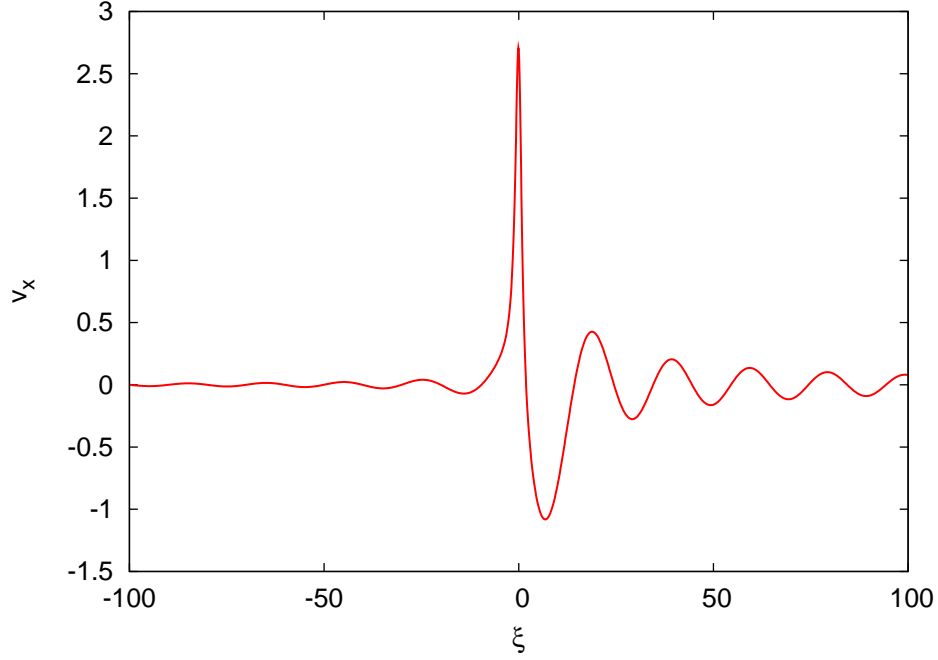


FIG. 3: Time dependence of the radial velocity $v_x(\xi)$.

Due to odd integrants $J_{c,g} = 0 = J_{s,u}$; $\lim_{Q \rightarrow 0}[QD_u] = 1.571$.

The formula for b_z describes mutual transformation between MHD waves with orthogonal polarization. Only for $K_z = 0$ we have exact separation between SMWs with xy -polarization and AWs with z -polarization.

An illustration for the initial conditions $\psi(-100) = 1$, $d_\xi \psi(-100) = 0$, $\chi(-\infty) = 0 = d_\xi \chi(-\infty)$ and parameters $K_y = 0.3$, $K_z = 0.1$, is presented at the figures below. For reliability and check of formulae the figures are doubled by the numerical solution of the set Eq. (12) by Runge–Kutta method.²⁴ Those general formulas give a solution to the Cauchy problem. Having in the beginning $t = t_0$ a distribution of the magnetic field $\mathbf{B}_{\text{wave}}(\mathbf{r}, t_0)$ and velocity

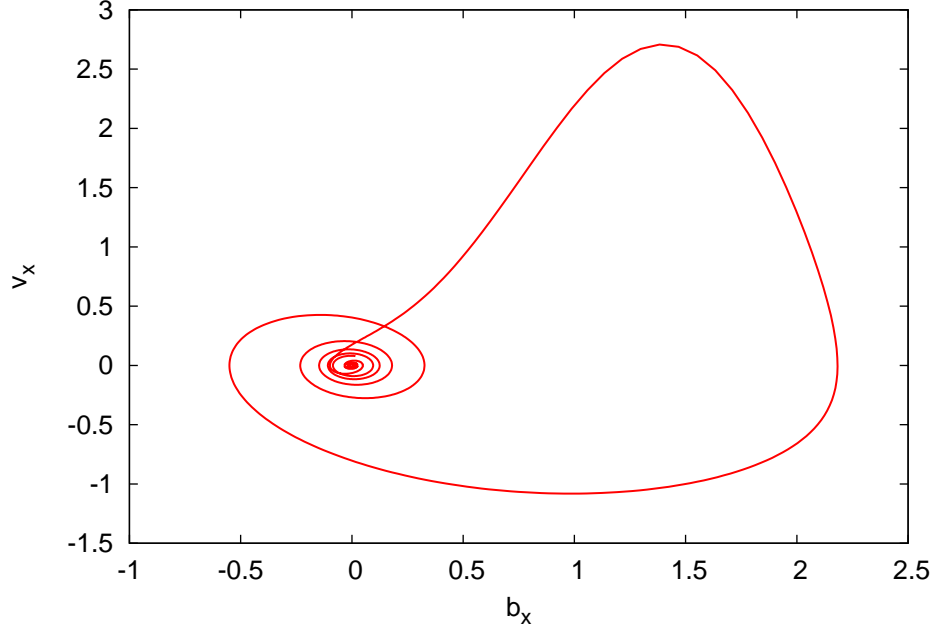


FIG. 4: Phase portrait of the radial $\hat{r} = \mathbf{e}_x$ motion v_x versus b_x . The big loop outlined at $\xi = 0$ is the driving force of SMW amplification. This kick amplifies magnetosonic waves, but the radial amplitude disappears at $\xi = 100$. The curve starts and ends at the point $(0,0)$.

with $\nabla \cdot \mathbf{V}_{\text{wave}}(\mathbf{r}, t_0) = 0$, we can calculate the Fourier components

$$\mathbf{v}_{\mathbf{k}}(t_0) = i \int \frac{\mathbf{V}_{\text{wave}}(\mathbf{r}, t_0)}{V_A} e^{-i\mathbf{k} \cdot \mathbf{r}} d^3x, \quad (145)$$

$$\mathbf{b}_{\mathbf{k}}(t_0) = \int \frac{\mathbf{B}_{\text{wave}}(\mathbf{r}, t_0)}{B_0} e^{-i\mathbf{k} \cdot \mathbf{r}} d^3x, \quad (146)$$

$$\xi_{0,\mathbf{k}} \equiv -\frac{k_x}{\sqrt{k_y^2 + k_z^2}}, \quad (147)$$

and initial dimensionless time $\xi_{0,\mathbf{k}}$. If $k_z = 0$ then $\text{sgn}(k_y)\xi_{0,\mathbf{k}} = \tau_{0,\mathbf{k}} = -k_x/k_y$. Then we have to determine the coefficients C in the general solutions for ψ Eq. (28) and χ Eq. (22) using the initial values at t_0

$$\mathbf{b}_{\mathbf{k}}(\xi) = C_g \mathbf{b}_g + C_u \mathbf{b}_u + \tilde{C}_g \mathbf{b}_{\tilde{g}} + \tilde{C}_u \mathbf{b}_{\tilde{u}}, \quad (148)$$

$$\mathbf{v}_{\mathbf{k}}(\xi) = C_g \mathbf{v}_g + C_v \mathbf{b}_u + \tilde{C}_g \mathbf{v}_{\tilde{g}} + \tilde{C}_u \mathbf{v}_{\tilde{u}}, \quad (149)$$

$$\mathbf{k} \cdot \mathbf{b}_{\mathbf{k}} = 0 = \mathbf{k} \cdot \mathbf{v}_{\mathbf{k}}. \quad (150)$$

In this set we can use only x - and z -components, and so we obtain 4 equation for the constants C_g , C_u , \tilde{C}_g , and \tilde{C}_u . The functions $\mathbf{b}_g(\xi)$ and $\mathbf{v}_g(\xi)$ are defined via substituting ψ_g in Eqs. (125)–(130) or Eqs. (131)–(142), and analogously ψ_u , χ_g , and χ_u . Then at each moment t we can calculate all variables in the \mathbf{k} -space

$$\xi_{\mathbf{k}}(t) = \xi_{0,\mathbf{k}} + (t - t_0) A \frac{k_y}{\sqrt{k_y^2 + k_z^2}}, \quad (151)$$

$$\mathbf{b}_{\mathbf{k}}(t) = \mathbf{b}_{\mathbf{k}}(\xi_{\mathbf{k}}(t)), \quad (152)$$

$$\mathbf{v}_{\mathbf{k}}(t) = \mathbf{v}_{\mathbf{k}}(\xi_{\mathbf{k}}(t)), \quad (153)$$

$$k_x^{\text{wave}}(t) = k_x - (t - t_0) A k_y. \quad (154)$$

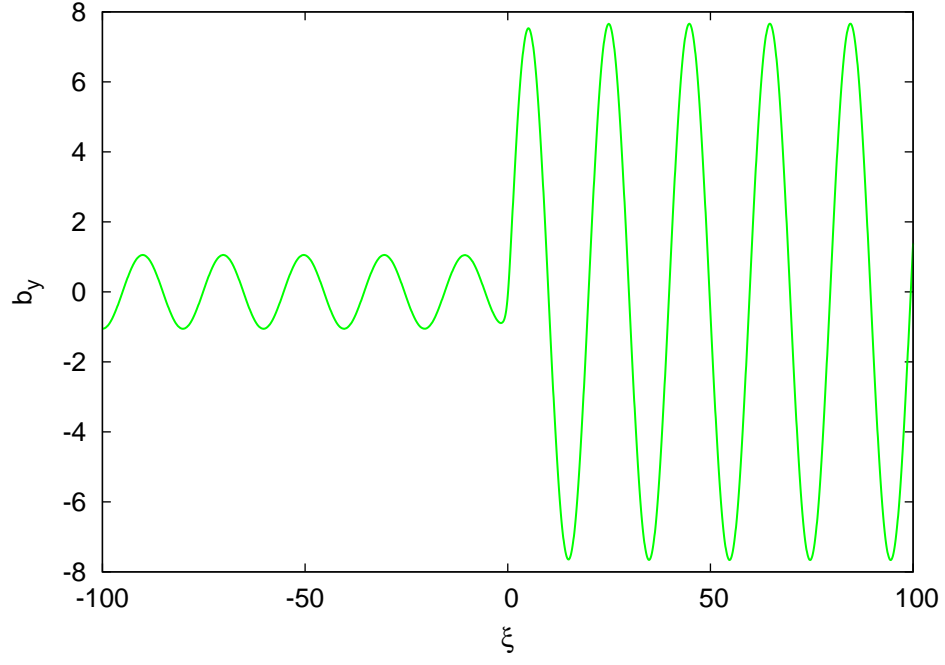


FIG. 5: Wave component of the magnetic field parallel to the constant one $B_0 = B_0 e_y$ as function of the dimensionless time $b_y(\xi)$.

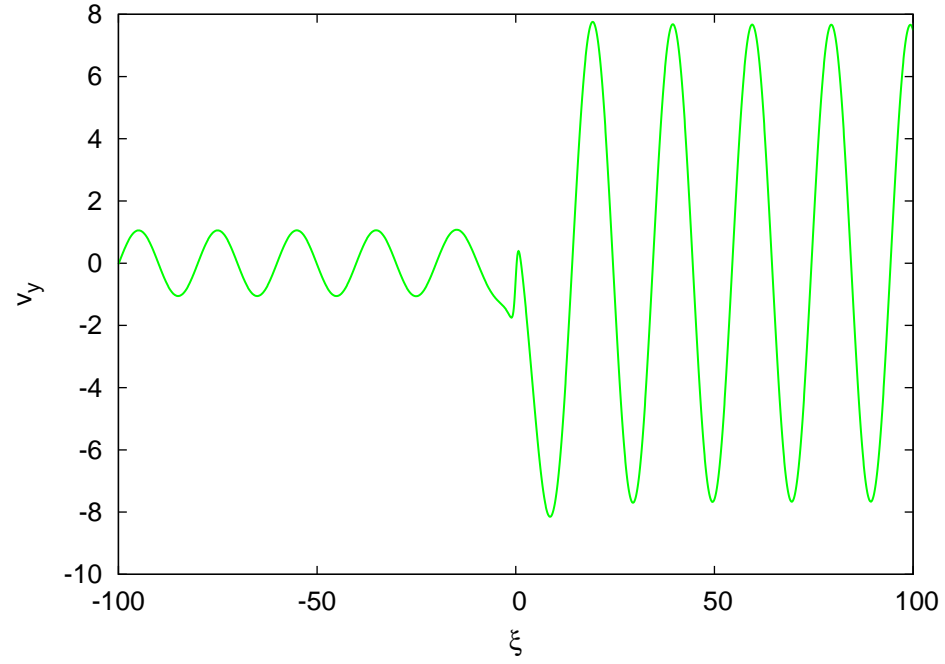


FIG. 6: Longitudinal with respect to the magnetic field B_0 velocity oscillations as function of the time $v_y(\xi)$.

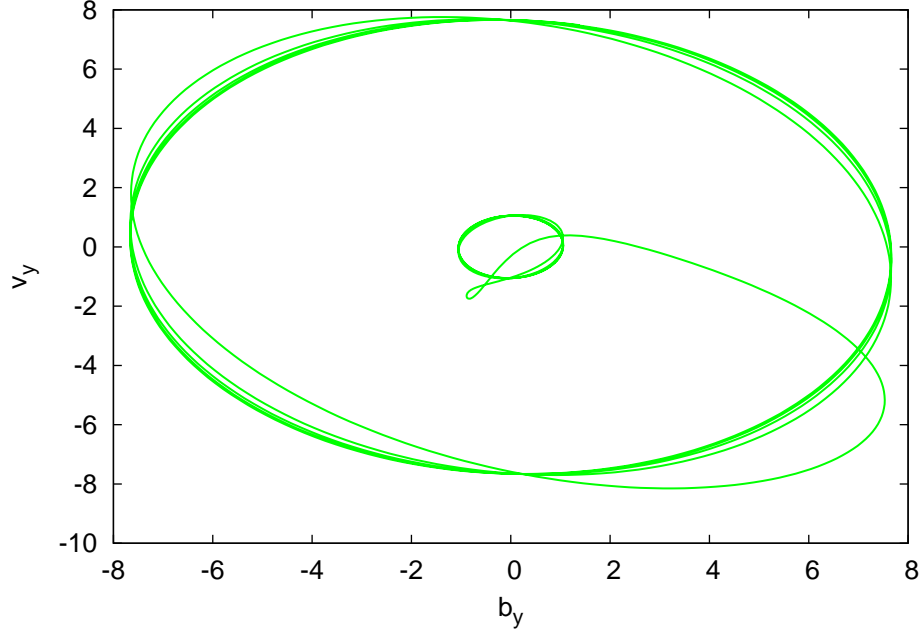


FIG. 7: Phase plot v_y versus b_y of axial $\hat{\varphi} = \mathbf{e}_y$ SMW oscillations parallel to the constant magnetic field $B_0 \hat{\varphi}$. The wave amplification is the ratio of the cycle areas at $\xi \rightarrow \infty$ and $\xi \rightarrow -\infty$. The fast transition between those two orbits describes the lazing of the alfvénons.

Finally, we can return back to the real \mathbf{r} -space

$$\begin{aligned} \mathbf{V}_{\text{wave}}(\mathbf{r}, t) &= -iV_A \int \mathbf{v}_{\mathbf{k}}(t) e^{-i[\mathbf{k} \cdot \mathbf{r} - (t-t_0)Ak_y x]} \frac{dk^3}{(2\pi)^3}, \\ \mathbf{B}_{\text{wave}}(\mathbf{r}, t) &= B_0 \int \mathbf{b}_{\mathbf{k}}(t) e^{-i[\mathbf{k} \cdot \mathbf{r} - (t-t_0)Ak_y x]} \frac{dk^3}{(2\pi)^3}, \\ \text{Re}(e^{-i\varphi}) &= \cos \varphi, \quad \text{Re}(-ie^{-i\varphi}) = -\sin \varphi. \end{aligned} \quad (155)$$

This evolution of MHD variables is the main detail of the theory of MHD turbulence in a shear flow.

Consider now an imaginary fluid filling the phase space \mathbf{k} and $w(t) \equiv \mathcal{E}_{\mathbf{k}}(t)$ from Eq. (124) being the energy density carried by a droplet of that fluid. As a wave mode initially with wave-vector \mathbf{k} evolves according to Eq. (122), the infinitesimal phase-fluid droplet associated with that mode moves in the \mathbf{k} -space. Wave amplification means that the energy density of the droplets increases by a factor of G

$$G = \frac{w(t \rightarrow +\infty)}{w(t \rightarrow -\infty)}. \quad (156)$$

Indeed for $\chi = 0$, $k_z = 0$, and $\xi \rightarrow \infty$

$$b_y^2 \asymp \psi^2 \gg b_x^2 + b_z^2, \quad v_y^2 \asymp \left(\frac{d_\xi \psi}{Q} \right)^2 \gg v_x^2 + v_z^2 \quad (157)$$

and

$$w(t \rightarrow \infty) = \frac{1}{4} V_A^2 D_f^2, \quad w(t \rightarrow -\infty) = \frac{1}{4} V_A^2. \quad (158)$$

For big enough time arguments $|\xi| \gg 1$ and purely two-dimensional waves with $k_z = 0$ the motion of the fluid asymptotically corresponds to a SMW with dispersion coinciding with the AW one

$$\begin{aligned} Q\xi &= \omega_{\text{SMW}} t, \quad \omega_{\text{SMW}} = \omega_{\text{AW}} = V_A |k_y|, \\ \psi(\xi) &\asymp D_f \cos(\omega_{\text{SMW}} t + \phi_f). \end{aligned} \quad (159)$$

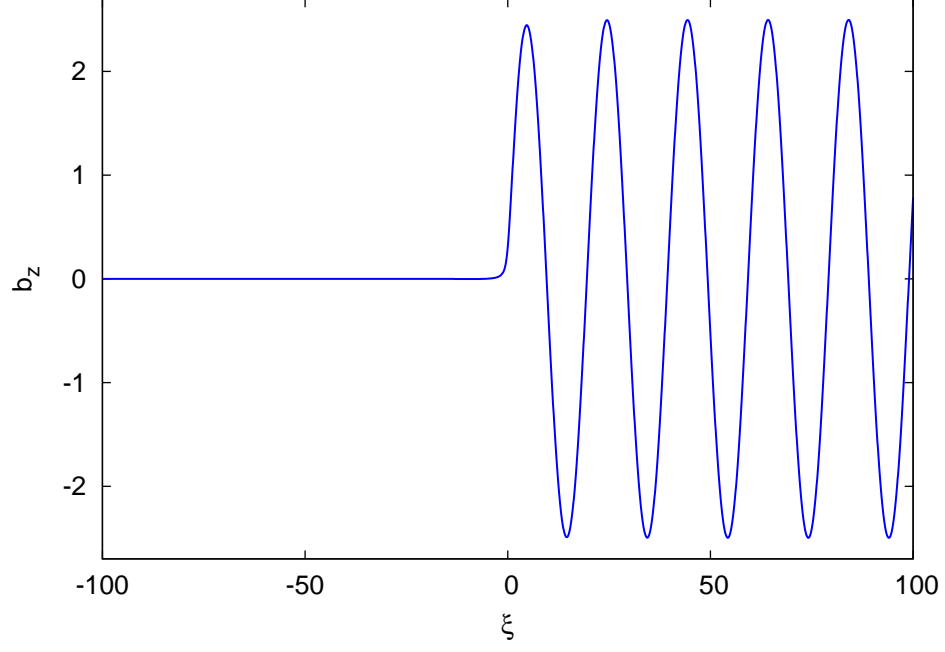


FIG. 8: Time dependence of the magnetic field oscillations $b_z(\xi)$ perpendicular to the disk plane.

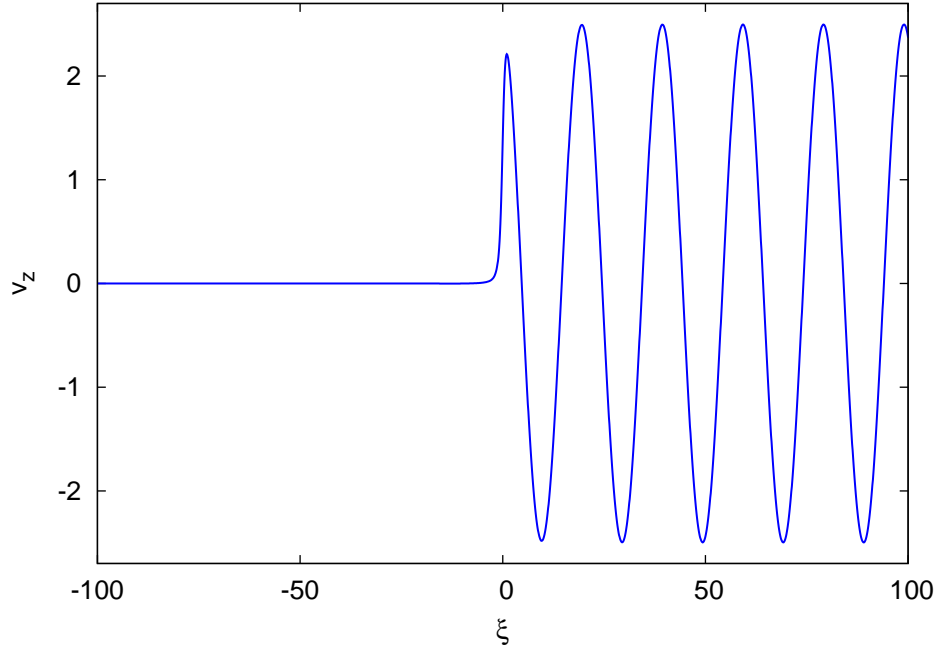


FIG. 9: Out of plane velocity oscillations parallel to the disk axis \hat{z} as function of the dimensionless time $v_z(\xi)$.

The Poynting vector, i.e., the energy flux in \mathbf{r} -space is $V_A w$.

The velocity of the droplet in the \mathbf{k} -space according to Eq. (122) determines the field of the shear flow in the \mathbf{k} -space

$$\mathbf{U} = d_t \mathbf{q}(t) = -A q_y \mathbf{e}_x, \quad \mathbf{U}_{\mathbf{k}}^{\text{shear}} = -A k_y \mathbf{e}_x. \quad (160)$$

Looking at a droplet we actually derive the shear flow velocity field in \mathbf{k} -space, $\mathbf{U}_{\mathbf{k}}^{\text{shear}}$.

According to the Kolmogorov–Obukhov cascade of energy we have a constant energy flux through each spherical

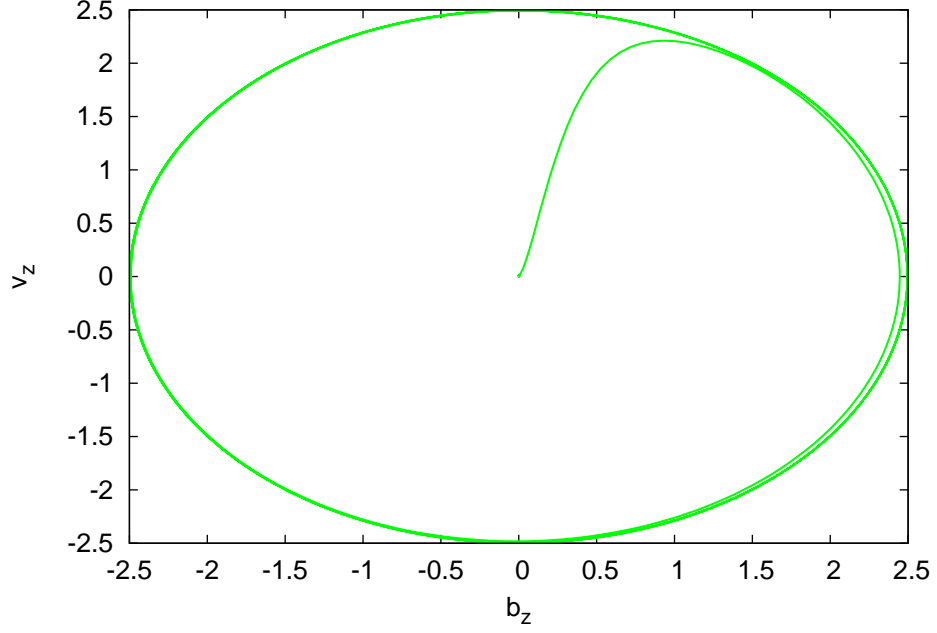


FIG. 10: Phase portrait of the AW oscillations transversal to the disk plane; v_z versus b_z . The AW polarization is almost perpendicular the constant component of the magnetic field and the wavevector for $\xi = 100$. The wave amplitude at $\xi = -100$ is negligible; we have mode conversion of SMW to AW at $\xi = 0$.

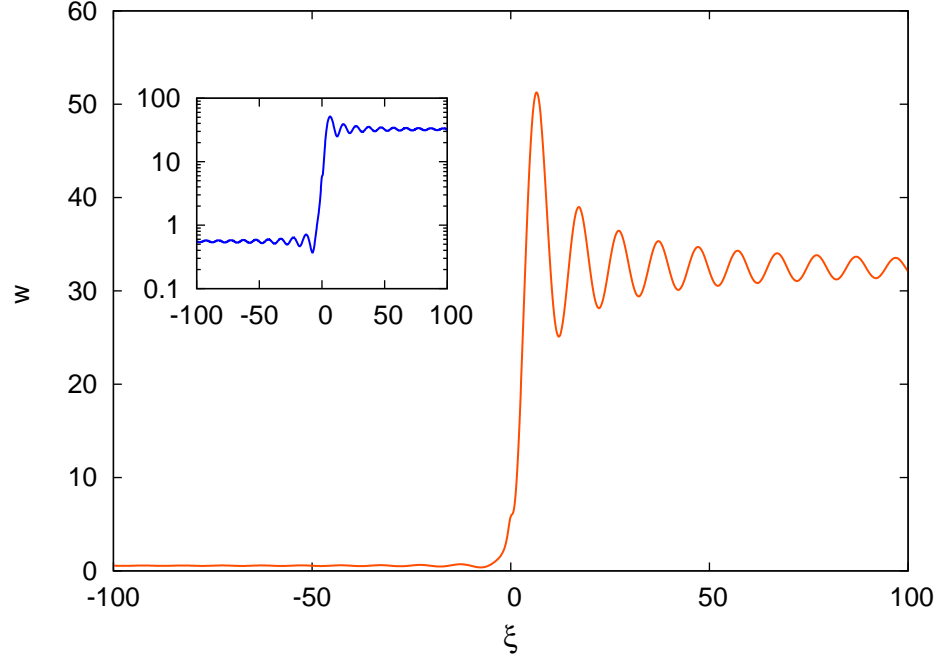


FIG. 11: Dimensionless wave energy density $w = \frac{1}{2}(\mathbf{v}^2 + \mathbf{b}^2)$ versus dimensionless time ξ . In the inset the ordinate is logarithmic. The ratio $w(100)/w(-100)$ describes the energy amplification by the shear flow which is the heating mechanism of accretion disks. The analytical expression of the energy gain Eqs. (131–142) is given by the confluent Heun functions Eqs. (31, 32). The approximative central symmetry of the blue curve corresponds to 10% accuracy of the product $w(t)w(-t) \simeq \text{const}$ for this numerical example.

surface with surface element $d\mathbf{f}$ in \mathbf{k} -space

$$\varepsilon = \oint \mathcal{E}_{\mathbf{k}}^{\text{KO}} \mathbf{U}_{\mathbf{k}}^{\text{KO}} d\mathbf{f} = \mathcal{E}_k^{\text{KO}} U_k^{\text{KO}} 4\pi k^2, \quad (161)$$

which gives

$$\mathbf{U}_{\mathbf{k}}^{\text{KO}} \sim \varepsilon^{1/3} k^{5/3} \mathbf{e}_k, \quad \mathbf{e}_k = \frac{\mathbf{k}}{k}, \quad (162)$$

i.e., the velocity in \mathbf{k} -space has dimension $1/(\text{time} \times \text{length})$. Here we used an important for our further work notion of the energy flux in the \mathbf{k} -space

$$\mathbf{S} = \mathcal{E}_{\mathbf{k}} \mathbf{U}_{\mathbf{k}} \quad (163)$$

which is equal to energy density times velocity in the \mathbf{k} -space. This notion is analogous to the Poynting vector being, however, defined in the \mathbf{k} -space. In the Kolmogorov–Obukhov scenario we have

$$\frac{\partial}{\partial \mathbf{k}} \cdot \mathbf{S}^{\text{KO}} = \varepsilon \delta(\mathbf{k}). \quad (164)$$

In order to approximate the turbulence as an initial source of MHD waves we have to merge the turbulence with the wave spectral densities and velocities. The simplest possible scenario is given in the next subsection.

B. Derivation of Shakura–Sunyaev Phenomenology in the Framework of Kolmogorov Turbulence

How vortices create waves is a complicated problem far beyond the scope of the present study. Here we will give only a model illustration merging the spectral density of vortices $\mathcal{E}_{\mathbf{k}}^{\text{turb}}$ from Kolmogorov turbulence with spectral density of magnetosonic waves $\mathcal{E}_{\mathbf{k}}^{\text{wave}}$

$$\mathcal{E}_{\mathbf{k}}^{\text{wave}} \sim \mathcal{E}_{\mathbf{k}}^{\text{turb}} \sim \mathcal{E}_{\Lambda} = \varepsilon^{2/3} \Lambda^{11/3}, \quad (165)$$

on the plane in momentum space

$$k_x = -\text{sgn}(k_y) \Lambda^{-1}, \quad k_y^2 + k_z^2 < \Lambda^{-2}, \quad (166)$$

where we qualitatively suppose that vortices are converted into waves. Sign function corresponds to the direction of the shear flow in the \mathbf{k} -space, Eq. (160). For k_y we consider that turbulent vortices have a given spectral density at $k_x > \Lambda^{-1}$ which is converted to MHD wave energy at the plane $k_x = \Lambda^{-1}$, and further on this wave energy evolves according to our solution. In other words, the plane $k_x > \Lambda^{-1}$ is the boundary between the vortex region and the beginning of the amplification in the wave region where vortices have negligible influence. In our qualitative picture we suppose that vortices create spectral density which further on evolves as wave spectral density with negligible influence.

The amplification Eq. (110) is essential $\mathcal{G} \gg 1$ only within a cylinder

$$\mathcal{G}(k_y, k_z) - 1 \sim \frac{1}{\Lambda^2 q^2}, \quad q^2 = k_y^2 + k_z^2 < \Lambda^{-2} \quad (167)$$

with radius Λ^{-1} . This result with remains unchanged in amplitude if we include the $J_{\text{c,u}}$ and $J_{\text{s,g}}$ terms.

The amplification occurs in the region $-\Lambda^{-1} < k_x < \Lambda^{-1}$, that is to say from the cylinder we cut a tube with length $2\Lambda^{-1}$. In other words, we have a domain with a shape of a tube in momentum space

$$\mathcal{V} = \{k_y^2 + k_z^2 < \Lambda^{-2}, \quad |k_x| < \Lambda^{-1}\}. \quad (168)$$

In order to calculate the total power of waves \mathcal{H} (per unit mass) analogously to Eq. (161) we will integrate the energy flux on the surface of the tube

$$\mathcal{H} = \oint \mathcal{E}_{\mathbf{k}}^{\text{wave}} \mathbf{U}_{\mathbf{k}}^{\text{shear}} d\mathbf{f}. \quad (169)$$

As the shear in the physical flow results in a drift of the wave modes along the axis of the tube, we have to take into account only the circular surfaces

$$\epsilon \sim \int |U_x| [\mathcal{G}(k_y, k_z) - 1] \mathcal{E}_\Lambda dk_y dk_z. \quad (170)$$

The multiplier $(\mathcal{G} - 1)$ takes into account the difference between flowing out and flowing in energy fluxes.

We can use polar coordinates

$$k_z = q \cos \theta, \quad k_y = q \sin \theta, \quad U_x = Aq \sin \theta. \quad (171)$$

Averaging over the angle θ

$$\langle U_x \rangle = \frac{2}{\pi} Aq \sim Aq, \quad \int_0^\pi \sin \theta \frac{d\theta}{\pi} = \frac{2}{\pi}, \quad (172)$$

and substituting it in Eq. (169), using $dk_y dk_z = d(\pi q^2)$, leads to the simple integral

$$\mathcal{H} \sim \int_0^{\Lambda^{-1}} \frac{Aq}{\Lambda^2 q^2} \mathcal{E}_\Lambda q dq \sim \frac{A\mathcal{E}_\Lambda}{\Lambda^3} \sim AV_\Lambda^2. \quad (173)$$

Then for the volume density of the amplified waves we have

$$\mathcal{Q} \equiv \rho \mathcal{H} \sim \rho AV_\Lambda^2 \sim \rho (\varepsilon V_A)^{2/3} A^{1/3}. \quad (174)$$

As all waves are finally dissipated, \mathcal{Q} is actually the volume density of plasma heating.

For evanescent Kolmogorov turbulence power

$$\rho \varepsilon \ll A \rho V_A^2 = AB_0^2 / \mu_0 \quad (175)$$

the heating power \mathcal{H} has a critical behavior

$$d_\varepsilon \mathcal{H} \sim G_{\text{turb}} \equiv \frac{\mathcal{H}}{\varepsilon} \sim \left(\frac{AV_\Lambda^2}{\varepsilon} \right)^{1/3} \gg 1, \quad \mathcal{H} \gg \varepsilon \quad \text{for } \varepsilon \rightarrow 0 \quad (176)$$

which demonstrates that disks can ignite as a star even for very weak turbulence and magnetic field. The ratio of wave power and Kolmogorov vortex power G_{turb} can be considered as an amplification coefficient for the turbulence. This energy gain shows how efficient is the transformation of shear flow energy into waves or in a broader framework the transformation of gravitational energy into heat of accretion disks.

For hydrogen plasma $\rho c_s^2 / p = 5/3 \sim 1$. Now we can evaluate the shear stress (as given by the ratio of the volume density of heating power and the shear frequency)

$$\sigma = \frac{2\mathcal{Q}}{A} \sim \rho \left(\frac{\varepsilon V_A}{A} \right)^{2/3} \sim \rho V_\Lambda^2 \quad (177)$$

via an effective viscosity

$$\eta_{\text{eff}} = \frac{\sigma}{A} \sim \rho \frac{(\varepsilon V_A)^{2/3}}{A^{5/3}}, \quad \nu_{\text{eff}} = \frac{\eta_{\text{eff}}}{\rho} \sim \frac{\mathcal{H}}{A^2} \sim \frac{(\varepsilon V_A)^{2/3}}{A^{5/3}} \quad (178)$$

and the dimensionless Shakura–Sunyaev friction coefficient

$$\alpha \equiv \frac{\sigma}{p} \sim \frac{V_\Lambda^2}{c_s^2}. \quad (179)$$

Including of the energy of z -polarized AWs does not modify this result. Here we wish to emphasize that in our evaluation of the energy gain, we were concentrated on the wave amplification of the energy of two dimensional motion in the x – y plane. Taking into account the energy in z -direction shows that the heating is even higher, which is of course in the favor of the concepts.

For an approximately Keplerian disk rotation the shear rate is half of the frequency of the orbital Keplerian angular velocity $A = -\frac{1}{2}\omega_{\text{Kepler}}$. In this case for time A^{-1} the disk rotates per 2 radians. For Earth's rotation along the

Sun this time is of the order of one season. In such a way the length parameter of our problem $\Lambda = V_A/A$ can be evaluated as one Alfvén season. Then V_A from the final result for the Shakura–Sunyaev parameter can be qualitatively considered as a pulsation of the turbulent velocity for two disk particles at distance equal to one Alfvén season Λ . Our theory is formally applicable for $V_A \ll c_s$ but the boundary of its applicability (when compressibility effects stop the SMWs amplification) allows us to understand that strong disk’s turbulence can lead to Shakura–Sunyaev upper limit $\alpha \sim 1$. Thus the following cascade of events emerges as a likely scenario for the intense heating in accretion flows: the heating of the bulk of the disk creates convection. For strong heating the convection is turbulent. Turbulence generates magnetohydrodynamic waves. Waves are amplified by the shear flow – this is the transformation of gravitational energy of orbiting plasma into waves. Waves finally are absorbed by the viscosity which creates the heating. The heat is emitted through the surface of the disk. This process of formation of stars and other compact astrophysical objects from nebulae works continuously – we have a self-consistent theory for self-sustained turbulence of the magnetized accretion disks.

The weak point of this scenario is the supposed convective turbulence which in presence of magnetic fields is unlikely to be of Kolmogorov type. We consider as much more plausible scenario the appearance of a self-sustained magneto-hydrodynamical turbulence considered in the next subsection.

C. Kraichnan Turbulence as a more Plausible Scenario for Accretion Disks

Magnetic field qualitatively changes the behavior of the fluid. We have no waves generated by vortices – the turbulence in magnetic field is related to MHD waves. Analogously to the Kolmogorov law Eq. (119), for the Kraichnan turbulence the power of energy cascade in the dissipation-free regime is given by the wave–wave interaction

$$\varepsilon = \frac{(V_\lambda^2)^2}{\lambda V_A} = \frac{(\text{velocity})^3}{\text{length}} = \frac{\text{power}}{\text{mass}}. \quad (180)$$

This power is proportional to the intensity of the two interacting waves and this nonlinear effect for incompressible fluid is due to the convective term $\mathbf{V} \cdot \nabla \mathbf{V}$ of the substantial acceleration $D_t \mathbf{V} = \partial_t \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V}$ of the momentum equation Eq. (8).

The theory of generation of SMWs invokes parallels with the nonlinear optical phenomena in lasers. The velocity oscillations of two amplified MHD waves $\mathbf{V}^{(a)}$ and $\mathbf{V}^{(b)}$ create an external driving force of the new wave with velocity field \mathbf{V} . In the linearized Eq. (8) we have to insert a small nonlinear correction

$$\rho \partial_t \mathbf{V} = -\nabla p + \frac{\nabla \times \mathbf{B}}{\mu_0} \times \mathbf{B} + \rho \mathbf{f}, \quad \mathbf{f} \equiv \frac{1}{2} \sum_{a,b} \mathbf{V}^{(a)} \cdot \nabla \mathbf{V}^{(b)}. \quad (181)$$

Here, in the inhomogeneous term \mathbf{f} we have to perform summation over all other MHD waves. This external for the wave force (per unit mass) acts as an external noise and its statistical properties are determined by the force–force correlator

$$\hat{\Gamma}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \langle \mathbf{f}(\mathbf{r}_1, t_1) \mathbf{f}(\mathbf{r}_2, t_2) \rangle, \quad (182)$$

where the averaging is over the waves phases. A scenario of such type (a Langevin MHD) was described in Ref. [15]; this approach is similar in the spirit to the forced burgers turbulence.²⁵ In the framework of that scenario the strongly amplified $|D_f| \gg 1$ MHD waves with asymptotics Eq. (73)

$$\psi \approx D_f \theta(\xi) \cos(Q\xi + \phi_f) \exp(-\nu' K_y^2 \tau^3 / 6) \quad (183)$$

generate new waves and after a statistical averaging we have a self-consistent theory for magnetic turbulence in a shear flow. So MHD waves ignite the chain reaction of quasar self-heating. The last exponential term describes the wave damping when a small viscosity is taken into account. Damping is significant only for $t \rightarrow \infty$ when $|k_x| \gg |k_y|$ and the wave-vector is almost parallel to the magnetic field. In this geometry, the damping rate of the wave density of AWs and SMWs is proportional to the square of the frequency¹⁷

$$w(t) = w(0) \exp\left(-\frac{\omega^2}{V_A^2} \nu t\right) = w(0) \exp(-\nu k^2 t). \quad (184)$$

For the time-dependent wave-vector $k^2(t) \approx (k_y A t)^2$ in the argument of the exponent we have to make the replacement

$$\nu k^2 t \rightarrow \nu \int_0^t k^2(t') dt' = \frac{1}{3} \nu' K_y^2 \tau^3, \quad \nu' = \frac{\nu A}{V_A^2} = \frac{1}{\mathcal{R}}, \quad \mathcal{R} \equiv \frac{\Lambda V_A}{\nu_k}, \quad \mathcal{S}_{\text{MRI}} \equiv \frac{\Lambda V_A}{\nu_{\text{m}}}, \quad (185)$$

where ν' is the dimensionless viscosity and $\mathcal{R} = V_A^2/A$ is the “Reynolds number of the magnetorotational instability (MRI),²⁶” and \mathcal{S}_{MRI} is the Lundquist number of MRI. After long enough time t_ν when

$$|k_x(t_\nu)| = 1/\lambda_a, \quad \lambda_a = \frac{\nu}{V_A}, \quad (186)$$

MHD waves are completely dissipated.

The details of self-consistent MHD turbulence will be given elsewhere, but again the wave amplification operates as a turbulence amplifier. For MHD turbulence one can expect

$$\sigma_{R\varphi} = \alpha_m(\nu')p_B, \quad p_B = \frac{1}{2}\rho V_A^2. \quad (187)$$

The evaluation of the magnetic friction coefficient α_m as a function of the dimensionless viscosity is a new problem addressed to the theoretical astrophysics.

V. DISCUSSIONS, CONCLUSIONS, AND PERSPECTIVES

We propose that the long-missing element of the dissipation mechanism of accretion disks is now identified – it is the amplification of long-wavelength SMWs. This is the indispensable ingredient of the stars generating engine. Without it the Universe would possibly be a structureless gas – deserted and uninhabited. The two-dimensional disks and the redistribution of angular momentum in them are the means of creating a one-dimensional compact astrophysical object from gases and dust. The planetary system is the result of this star-producing sequence of events when the accretion stops and the disk is frozen. It is remarkable that a simple equation of the Schrödinger type, Eq. (27), is at the core of the friction mechanism which cheated the diversity of the Universe. This means that the Schrödinger equation can describe one more phenomenon while science is on its way to explaining the frogs and the musical composers. “The next great era of awakening of human intellect may well produce a method of understanding the qualitative content of equations. Today we cannot. Today we cannot see that the water flow equations contain such things as the barber pole structure of turbulence that one sees between rotating cylinders. Today we cannot see whether Schrödinger’s equation contains frogs, musical composers, or morality – or whether it does not.”²⁷

Notably, here we have observed only an amplification in a weak magnetic field but not instability, where the amplitude of waves increases infinitely with time. Also, the rotation is found not relevant for this phenomenon. Therefore, regardless of similarity in spirit, our work is completely different from the research focused on rotational instabilities in strong magnetic fields. Our results point to an amplification but not to an instability. For sufficiently long wavelengths the amplification can be enormous but never infinite. Amplification means exponential increasing of the amplitude in the framework of linear theory.

After investigating the local dissipation and shear tension by the self-consistent statistical MHD method the corresponding numerical value of the α parameter can be incorporated in global models for accretion disks dynamics. The global models for accretion disks include also the problem of disk dynamo.²⁸ In order to create a self-consistent magnetic field $B_{0,\varphi}$ the accretion disks operate as radial inflow generators from plasma physics,²⁹ see also Ref. 30. In this broad program the local and detailed investigation of magnetosonic waves propagation in a homogeneous shear flow and magnetic field is only the first step in our understanding of the properties of space plasmas, i.e., the low-gradient approximation is the indispensable step in our understanding of accretion power in the Universe.³¹

The further development of the theory of MHD waves turbulence will give additional important details but even with what we know now, we get an insight into the workings of the star-creating engine. A very powerful, hence possibly dominant, mechanism of energy transformation in shear magnetohydrodynamic flows is identified, which is the amplification of Alfvén waves. They are also responsible for the heating of the solar corona¹⁹ and represent a quite common mechanism for heating of space plasmas in general. Here we wish to recall that shear flows are important for formation of toroidal magnetic field from poloidal one. The differential rotation²⁰ of the Sun is another shear flow which can lead to amplification of MHD waves. The convection excites MHD waves in almost toroidal magnetic fluxtubes. When buoyant magnetic fluxtubes reach photospheric surface the MHD waves can be significantly amplified.

Recently Alfvén waves of sufficient strength have been unambiguously observed in the chromosphere by Solar Optical Telescope onboard the Japanese *Hinode* satellite. Such Alfvén waves are energetic enough to accelerate the solar wind and to heat the quiet corona.³² In order to reach quantitative agreement of the theory it is necessary to merge the spectral density $\propto D_{\text{AW}}/\omega^n$ of these AW with the speed of the solar wind v_{wind} and coronal temperature T_p . The mission of the theoretical models is to give the simple relations between those experimentally accessible parameters $f(T_p, v_{\text{wind}}, D_{\text{AW}}) = 0$, cf. Ref. [33]. Heating of tokamak plasmas by magnetosonic waves is also a widely discussed issue.^{34,35} According our scenario the self-heated accretion discs are just working tokamaks.

The idea of accretion disks can be traced in the development of contemporary science.³⁶ While for Descartes (1644), Kant, and Laplace that idea was a creative mythology,³⁷ the Hubble Space Telescope has now delivered observational evidence for disks of ionized gases around massive black holes.³⁸ Observations of protoplanetary disks are discussed at the web-page³⁹ and its links.^{40,41,42}

It was only several decades after a horrific punishment for the expression of now mainstream views took place at the market place Campo dei Fiori⁴³ in Rome, when Galileo observed spots on the Sun and detected the Sun's rotation. Why the Sun is rotating so slowly (i.e., the problem of angular momentum dissipation) eventually becomes a focus issue in cosmogony.⁴⁴ The development of plasma physics determined that the molecular viscosity of plasma is far too small to account for the observed heating and that new ideas should be tested. The amplification of SMWs, an idea with the potential of explaining the enormous luminosity of quasars, was explored in the present work.

Pursuing further applications and proof of concepts, one could notice that the magnetosonic waves of magnetic turbulence can emit radio waves through the large surface of the accretion disk. Correlation between the radio waves and optical emission from quasars will be the crucial test for the present theory of dissipation in accretion disks.

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